

Mathematics for the Liberal Arts

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Lumen Learning

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HISTORICAL COUNTING SYSTEMS

WHY IT MATTERS: HISTORICAL COUNTING SYSTEMS

Why do historical counting systems still matter today?

When you check that balance in your bank account, or when you glance at the speedometer in your car, or even when you look for your child's number on the back of jerseys during a pee wee football game, you are reading numerals in the **Hindu-Arabic counting system**. We are all familiar with those ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. What's more, when we read a number like 352, we know that it stands for three groups of a hundred, five groups of ten, and two units. Our numerals are arranged according to a **positional base 10 (or decimal) system**... most of the time, anyway.



Telling time requires a slightly different system. While there are still Hindu-Arabic numerals involved, the way that they behave is decidedly different. There are 60 seconds in every minute and 60 minutes in every hour. So if your watch displays 10:04:59 right now, then you expect it to read 10:05:00 a second later.

We are so used to telling time in groups of 60 that it seems natural. But have you ever wondered why there are not 100 seconds in each minute, or 100 minutes in an hour? In the late 1700s, a French attorney by the name of Claude Boniface Collignon suggested a system of decimal time measurement in which each day has 10 hours, each hour lasting 100 minutes, and each minute having 1000 seconds. Of course the actual duration of these new hours, minutes, and seconds would be much different. In particular, the decimal second, would last 0.864 of a normal second. On the upside, time conversions would be trivial; for example, 6 decimal hours = 600 decimal minutes = 600,000 decimal seconds.

So why does our system of telling time not conform to the usual base 10 counting system that governs most other aspects of our life? Blame it on the Babylonians!



The Babylonians were one of the first cultures to develop a positional numeral system. However instead of having only 10 distinct numerals and groups in powers of 10, their system was based on groups and powers of 60 (which is called a **sexigesimal system**). The Babylonian system spread throughout most of Mesopotamia, but it eventually faded into history, allowing other number systems such as the Roman numerals and the Hindu-Arabic system to take its place.

On the other hand, there are still vestiges of the sexigesimal counting system in the way that we keep time as well as how we measure angles in degrees. There are 360 degrees in a full circle (and $360^\circ = 6 \times 60^\circ$). Furthermore, there are 60 arc minutes in one degree and 60 arc seconds in one arc minute. This system of degrees, arc minutes, and arc seconds is also used to locate any point on the surface of the Earth by its latitude and longitude. So even though our numerals are Hindu-Arabic, we still rely on the Babylonian base 60 system every second of the day and everywhere on the globe!

Learning Objectives

Numeration systems

- Explore the counting and number system used by the Inca
- Become familiar with the evolution of our current counting method
- Convert between Hindu-Arabic and Roman Numerals

Positional Systems

- Become familiar with the history of positional number systems
- Identify bases that have been used in number systems historically
- Convert numbers between bases other than 10
- Use two different methods for converting numbers between bases

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INTRODUCTION: EARLY COUNTING SYSTEMS

As we begin our journey through the history of mathematics, one question to be asked is “Where do we start?” Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities. (Note: Eves, Howard; An Introduction to the History of Mathematics, p. 9.)

Where to start the study of the history of mathematics...

- At the first logical geometric “proofs” traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has *always* existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

Recognition of More vs. Less

The idea of number and the process of counting goes back far beyond history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago. (Note: Eves, p. 9.) However, we do not really know how this process started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover’s nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food. (Note: McLeish, John; The Story of Numbers—How Mathematics Has Shaped Civilization, p. 7.) With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

Learning Objectives

In this lesson you will:

- Determine the number of objects being represented by pebbles placed on an Inca counting board.
- Determine the number represented by a quipu cord
- Identify uses other than counting for a quipu cord
- Become familiar with the evolution of the counting system we use every day
- Write numbers using Roman Numerals
- Convert between Hindu-Arabic and Roman Numerals

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ENTERING ANSWERS FOR PRACTICE PROBLEMS

Welcome to Math for Liberal Arts. On each page of this interactive text, you will find opportunities to practice skills and gain feedback instantly. The following practice problems are intended to help you understand how to enter your answers for these types of activities.

Typically, you will see practice problems in green text boxes like the one below. In this problem, you are asked to enter a numerical answer. Read the introduction carefully so you don't get frustrated later.

Try It Now

Visit this page in your course online to practice before taking the quiz.

You may have noticed the prompt above the question that says "Try another version of this question." Most of the practice questions in this text are randomized so that when you click on the link, you will get another version of the question with new numbers. These problems are intended for practice, so try them as many times as you like. No scores are kept, and there's never a due date :).

In the next sample problem, you are given an opportunity to interact with the embedded equation editor. Some people find it easier to use the editor for things like square roots, fractions and exponents. Give it a try!

Try It Now

Visit this page in your course online to practice before taking the quiz.

Sometimes you will be asked to enter a mathematical expression or equation that contains things like square roots, fractions, or exponents. The next practice problem will give you a chance to try it out.

Try It Now

Visit this page in your course online to practice before taking the quiz.

In our last practice problem, we give you an opportunity to test-drive the graphing capability of the assessment engine. You are asked to plot ordered pairs on a coordinate plane. Don't worry if you don't remember how to do this, we give you instructions below.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Remember, the practice problems in this text are intended for you to try as many times as you like. No one is keeping track! Practice as much as you need.

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THE EVOLUTION OF COUNTING AND THE INCA COUNTING SYSTEM

The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy's camp. Certainly it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. "Just how many of them do we have, anyway?" is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with "tally sticks." These are objects used to track the numbers of items to be counted. With this method, each "stick" (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of **one to one correspondence**. In a one to one correspondence, items that are being counted are uniquely linked with some counting tool.

In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to keep track of. If a herder wanted to "count off" his animals to make sure they were all present, he could mentally (or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.

Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol “1,” the second stick gets replaced by a “2” and the third stick is represented by the symbol “3,” but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the “tally stick” counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absalom discovered a wolf bone that goes back possibly 30,000 years. It is believed to be a counting device. (Note: Bunt, Lucas; Jones, Phillip; Bedient, Jack; The Historical Roots of Elementary Mathematics, p. 2.) Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below. (Note: http://www.math.buffalo.edu/mad/Ancient-Africa/mad_zaire-uganda.html) It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.

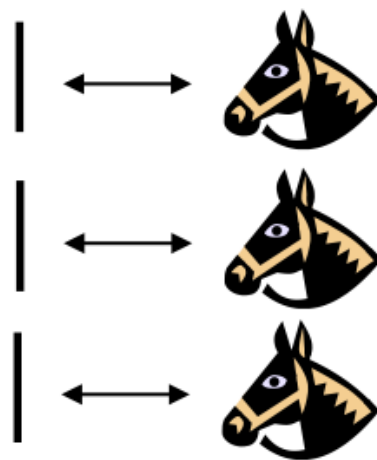


Figure 1.

The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.

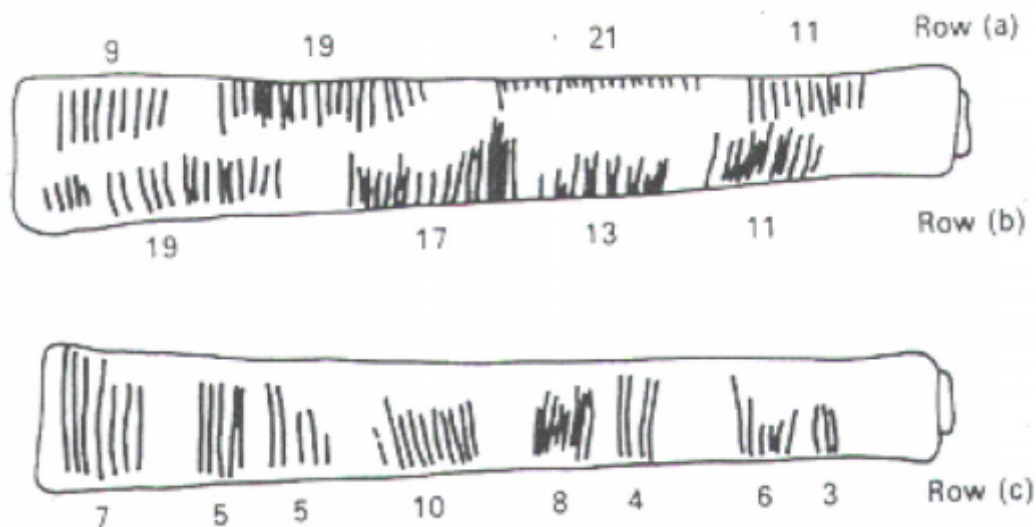


Figure 2.

Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

- Eleven comes from “ein lifon,” meaning “one left over.”
- Twelve comes from “twe lif,” meaning “two left over.”
- Thirteen comes from “Three and ten” as do fourteen through nineteen.
- Twenty appears to come from “twe-tig” which means “two tens.”
- Hundred probably comes from a term meaning “ten times.”

Written Numbers

When we speak of “written” numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is only a little more than 100 years old, so “writing” in times past often took on forms that might look quite unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be “read” by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which “writing” may have taken place include carvings in stone or clay tablets, rag paper made by hand (twelfth century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don’t have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the “important” information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded. (Note: Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623–28.) The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiöld, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the “quipu” to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight-sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.

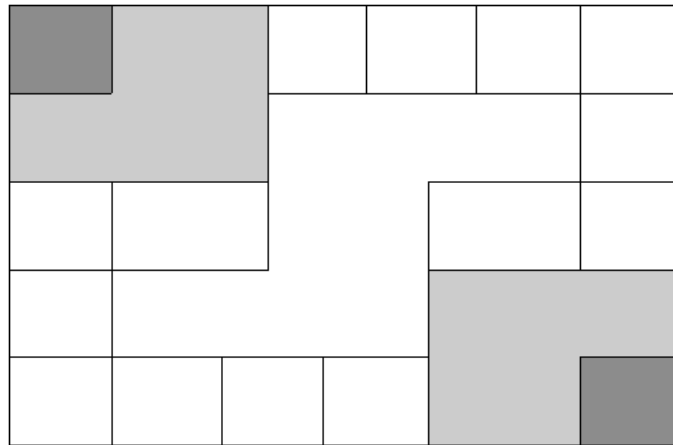
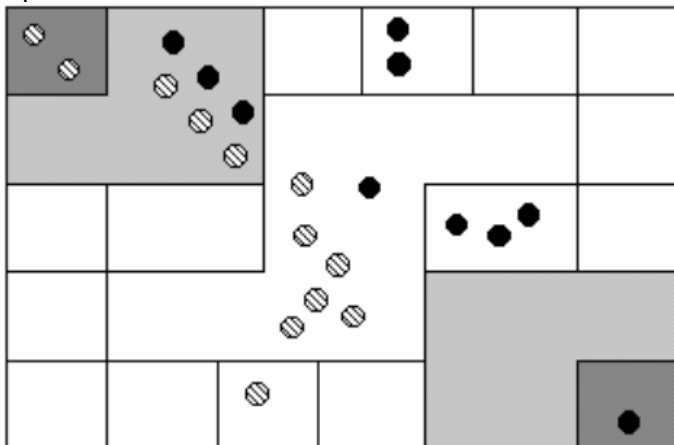


Figure 3.

Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?

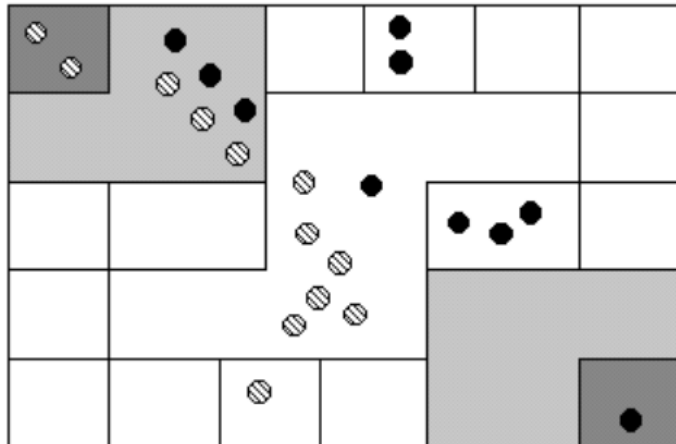


Answer

There are two black pebbles in the outer square regions...these represent 2 dogs. There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs. There is one black pebble in the middle region...this represents 3 dogs. There are three black pebbles on the second level...these represent 18 dogs. Finally, there is one black pebble on the highest corner level...this represents 12 dogs. We then have a total of $2+6+3+18+12 = 41$ dogs.

Try It Now

How many cats are represented on this board?



Answer

$$1+6'3+3'6+2'12 = 61 \text{ cats}$$

Watch this short video lesson about Inca counting boards. You will find that this is a review of concepts presented here about counting boards.

Watch this video online: https://youtu.be/fL1N_V89g78

The Quipu

This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right. (Note: Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623–28.)

Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a “totalizer” cord that summarizes all of the information on the cord group in one place.

Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of figure 6 (Note: <http://wiscinfo.doit.wisc.edu/chaysimire/titulo2/khipus/what.htm>) was used to denote tens, hundreds, thousands, and ten thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer “one.” Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represent tens and hundreds.) Note that the long knot has several turns in it...the number of turns indicates which integer is being represented. The

units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.

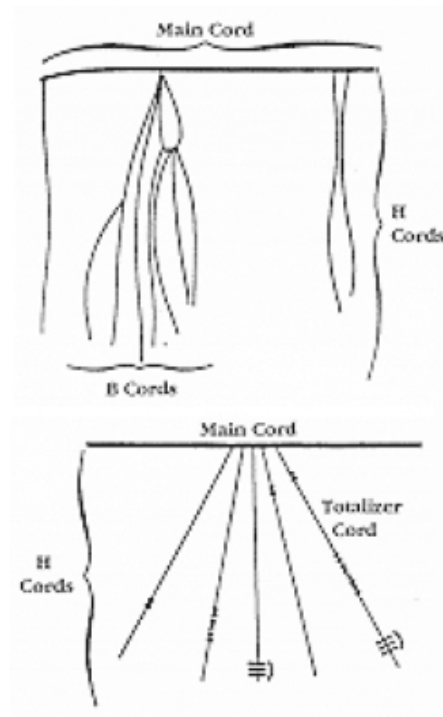


Figure 5.

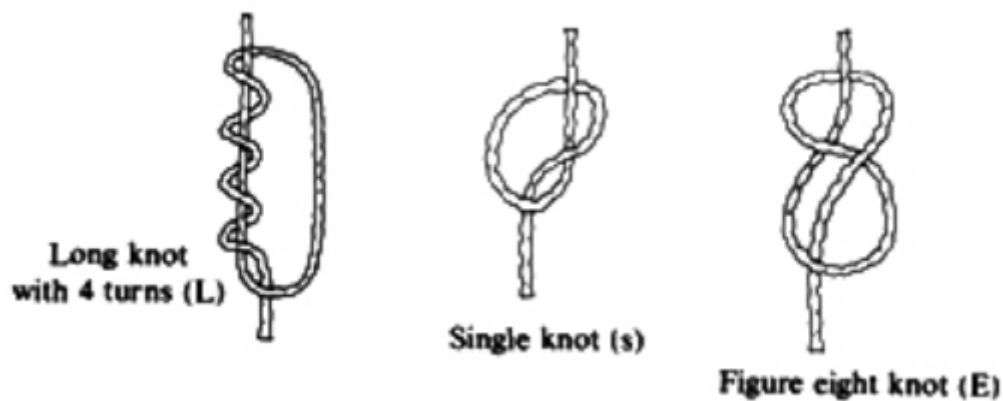


Figure 6

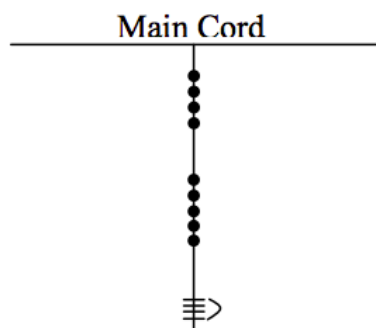
In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation:

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

We will represent the single knot with a large dot (\cdot) and we will represent the figure eight knot with a sideways eight (∞).

Example

What number is represented on the cord shown in figure 7?

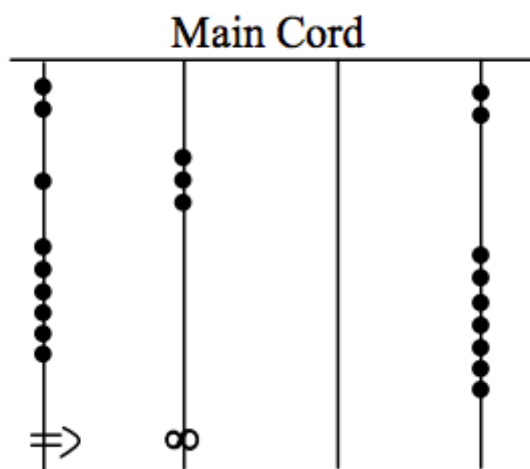


Answer

On the cord, we see a long knot with four turns in it...this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

Try It Now

What numbers are represented on each of the four cords hanging from the main cord?



Answer

From left to right:

Cord 1 = 2,162

Cord 2 = 301

Cord 3 = 0

Cord 4 = 2,070

The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re-used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.

As you can see from this photograph of an actual quipu (figure 9), they could get quite complex.

There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system. (Note: Diana, Lind Mae; The Peruvian Quipu in *Mathematics Teacher*, Issue 60 (Oct., 1967), p. 623–28.) Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent “conversion” to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

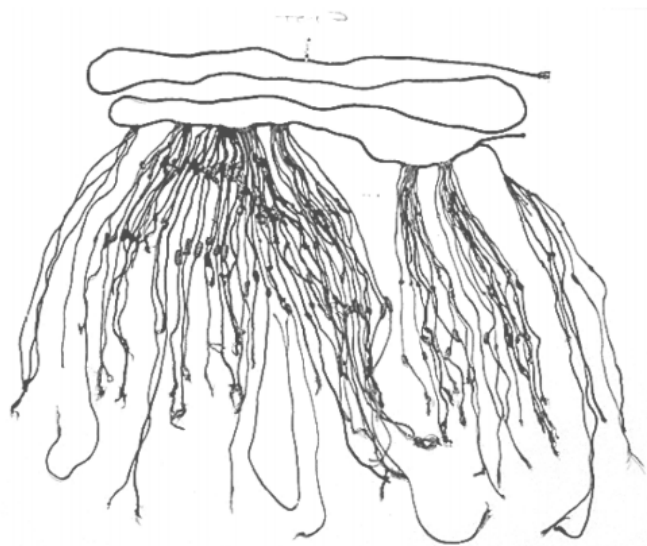


Figure 9.

The following video presents another introduction to the Inca’s use of a quipu for record keeping.

Watch this video online: <https://youtu.be/EYq-VtyAd2s>

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, *The Code of the Quipu: A Study in Media, Mathematics, and Culture*, which is “an extensive elaboration of the logical-numerical system of the quipu.” (Note:

<http://www.cs.uidaho.edu/~casey931/seminar/quipu.html>) For more information on the quipu, you may want to check out “Khipus: a unique Huarochiri legacy.”

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world. That’s not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

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THE HINDU–ARABIC NUMBER SYSTEM AND ROMAN NUMERALS

The Evolution of a System

Our own number system, composed of the ten symbols $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is called the *Hindu-Arabic system*. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India. (Note: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html)

It was not until the fifteenth century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al-Biruni, whose picture is shown in figure 10. (Note: <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Al-Biruni.html>) Al-Biruni, who was born in modern day Uzbekistan, had visited India on several occasions and made comments on the Indian number system. When we look at the origins of the numbers that al-Biruni encountered, we have to go back to the third century BCE to explore their origins. It is then that the Brahmi numerals were being used.



Figure 10. Al-Biruni

The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000, ..., also for 20, 30, 40, ..., and others for 200, 300, 400, ..., 900. The Brahmi symbols for 1, 2, and 3 are shown below. (Note: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html)

1	2	3
—	=	≡

Brahmi one, two, three

These numerals were used all the way up to the fourth century CE, with variations through time and geographic location. For example, in the first century CE, one particular set of Brahmi numerals took on the following form: (Note: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html)

1	2	3	4	5	6	7	8	9
—	=	≡	+	h	4	7	5	1

From the fourth century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the fourth through sixth centuries. They have the following form: (Note: Ibid.)

1	2	3	4	5	6	7	8	9
—	=	≡	५	६	७	८	९	०

How the numbers got to their Gupta form is open to considerable debate. Many possible hypotheses have been offered, most of which boil down to two basic types. (Note: Ibid.) The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon . . . the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this: (Note: Ibid.)

1	2	3	4	5	6	7	8	9
I	II	III	IIII	IIII	IIII	IIII	IIII	IIII

Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the eleventh century, at which time they looked like this: (Note: Ibid.)

1	2	3	4	5	6	7	8	9	0
१	२	३	४	५	६	७	८	९	०

Note that by this time, the symbol for 0 has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India. (Note: Katz, page 230) It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include: (Note: Burton, David M., *History of Mathematics, An Introduction*, p. 254–255)

1 2 3 4 5 6 7 8 9

Figure 11. Devangari, eighth century

1 2 3 4 5 6 7 8 9

Figure 12. West Arab Gobar, tenth century

1 2 3 4 5 6 7 8 9

Figure 13. Spain, 976 BCE

Finally, figure 14 (Note: Katz, page 231.) shows various forms of these numerals as they developed and eventually converged to the fifteenth century in Europe.

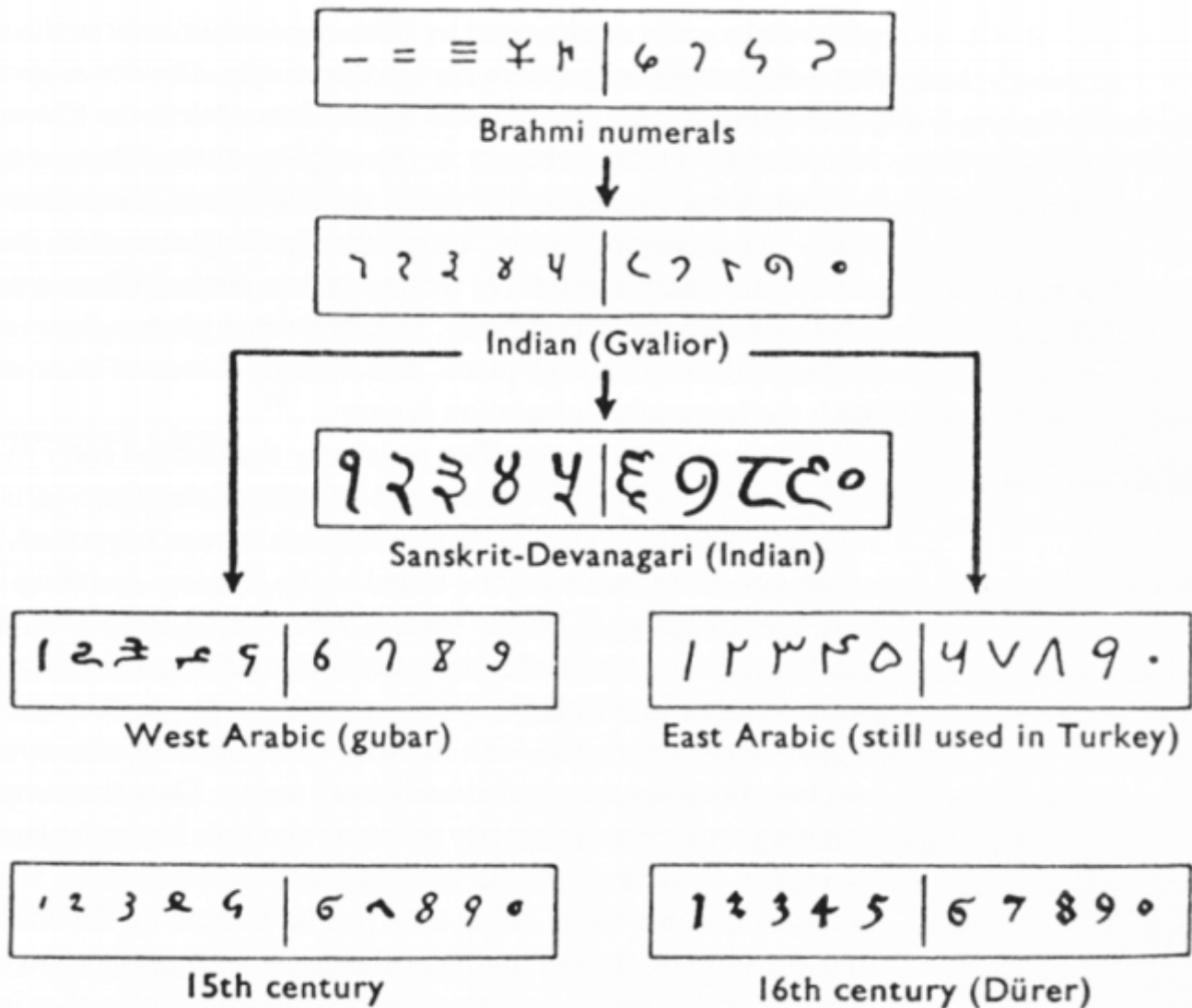


Figure 14.

Roman Numerals

The numeric system represented by **Roman numerals** originated in ancient Rome (**753 BC–476 AD**) and remained the usual way of writing numbers throughout Europe well into the Late Middle Ages (generally comprising the 14th and 15th centuries (c. 1301–1500)). Numbers in this system are represented by combinations of letters from the Latin alphabet. Roman numerals, as used today, are based on seven symbols:

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1,000

The use of Roman numerals continued long after the decline of the Roman Empire. From the 14th century on, Roman numerals began to be replaced in most contexts by the more convenient Hindu-Arabic numerals; however, this process was gradual, and the use of Roman numerals persists in some minor applications to this day.

The numbers 1 to 10 are usually expressed in Roman numerals as follows:

I, II, III, IV, V, VI, VII, VIII, IX, X.

Numbers are formed by combining symbols and adding the values, so II is two (two ones) and XIII is thirteen (a ten and three ones). Because each numeral has a fixed value rather than representing multiples of ten, one hundred and so on, according to *position*, there is no need for “place keeping” zeros, as in numbers like 207 or 1066; those numbers are written as CCVII (two hundreds, a five and two ones) and MLXVI (a thousand, a fifty, a ten, a five and a one).

Symbols are placed from left to right in order of value, starting with the largest. However, in a few specific cases, to avoid four characters being repeated in succession (such as IIII or XXXX), subtractive notation is used: as in this table:

Number	4	9	40	90	400	900
Roman Numeral	IV	IX	XL	XC	CD	CM

In summary:

- I placed before V or X indicates one less, so four is IV (one less than five) and nine is IX (one less than ten)
- X placed before L or C indicates ten less, so forty is XL (ten less than fifty) and ninety is XC (ten less than a hundred)
- C placed before D or M indicates a hundred less, so four hundred is CD (a hundred less than five hundred) and nine hundred is CM (a hundred less than a thousand)

Example

Write the Hindu-Arabic numeral for MCMIV.

Answer

One thousand nine hundred and four, 1904 (M is a thousand, CM is nine hundred and IV is four)

Try It Now

Visit this page in your course online to practice before taking the quiz.

Modern use

By the 11th century, Hindu–Arabic numerals had been introduced into Europe from al-Andalus, by way of Arab traders and arithmetic treatises. Roman numerals, however, proved very persistent, remaining in common use in the West well into the 14th and 15th centuries, even in accounting and other business records (where the actual calculations would have been made using an abacus). Replacement by their more convenient “Arabic” equivalents was quite gradual, and Roman numerals are still used today in certain contexts. A few examples of their current use are:



Spanish Real using “IIII” instead of IV

- Names of monarchs and popes, e.g. Elizabeth II of the United Kingdom, Pope Benedict XVI. These are referred to as regnal numbers; e.g. II is pronounced “the second”. This tradition began in Europe sporadically in the Middle Ages, gaining widespread use in England only during the reign of Henry VIII. Previously, the monarch was not known by numeral but by an epithet such as Edward the Confessor. Some monarchs (e.g. Charles IV of Spain and Louis XIV of France) seem to have preferred the use of IIII instead of IV on their coinage (see illustration).
- Generational suffixes, particularly in the US, for people sharing the same name across generations, for example William Howard Taft IV.
- In the French Republican Calendar, initiated during the French Revolution, years were numbered by Roman numerals – from the year I (1792) when this calendar was introduced to the year XIV (1805) when it was abandoned.
- The year of production of films, television shows and other works of art within the work itself. It has been suggested – by BBC News, perhaps facetiously – that this was originally done “in an attempt to disguise the age of films or television programmes.”^[23] Outside reference to the work will use regular Hindu–Arabic numerals.
- Hour marks on timepieces. In this context, 4 is usually written IIII.
- The year of construction on building faces and cornerstones.
- Page numbering of prefaces and introductions of books, and sometimes of annexes, too.
- Book volume and chapter numbers, as well as the several acts within a play (e.g. Act iii, Scene 2).
- Sequels of some movies, video games, and other works (as in *Rocky II*).
- Outlines that use numbers to show hierarchical relationships.
- Occurrences of a recurring grand event, for instance:
 - The Summer and Winter Olympic Games (e.g. the XXI Olympic Winter Games; the Games of the XXX Olympiad)
 - The Super Bowl, the annual championship game of the National Football League (e.g. Super Bowl XXXVII; Super Bowl 50 is a one-time exception^[24])
 - WrestleMania, the annual professional wrestling event for the WWE (e.g. WrestleMania XXX). This usage has also been inconsistent.

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INTRODUCTION: POSITIONAL SYSTEMS AND BASES

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 CE. However, some evidence suggests that they may have actually developed a positional system as far back as the first century CE.

In this lesson we will explore positional systems and their historical development. We will also discuss some of the positional systems that have been used throughout history and the bases used for those systems. Finally, we will learn how to convert numbers between bases and systems.

Learning Objectives

- Become familiar with the history of positional number systems
- Identify bases that have been used in number systems historically
- Convert numbers between bases
- Use two different methods for converting numbers between bases

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THE POSITIONAL SYSTEM AND BASE 10

The Indians were not the first to use a positional system. The Babylonians (as we will see in Chapter 3) used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems, except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board. (Note: Ibid, page 230) Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 CE, the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value. (Note: Ibid, page 231.) Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year of the Saka era is written with three digits and a dot in the middle. The 608th year uses

three digits with a modern 0 in the middle. (Note: Ibid, page 232.) The dot as a symbol for zero also appears in a Chinese work (*Chiu-chih li*). The author of this document gives a strikingly clear description of how the Indian system works:

Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus the numeral is always denoted in each place. Accordingly there can be no error in determining the place. With the numerals, calculations is easy. (Note: Ibid, page 232.)

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth-century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 CE and were there until 1492.

In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business. (Note: McLeish, p. 18) The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.

In a fourteenth century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 CE, however, the newer symbols and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu-Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years*. (Note: http://seattletimes.nwsources.com/news/health-science/html98/invs_20000201.html, Seattle Times, Feb. 1, 2000)

One question to answer is *why* the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics. (Note: Ibid, page 232.) Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.



Figure 15.

The Development and Use of Different Number Bases

Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is "positional." We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other

possible base systems. In the next part of this section, we will journey back to Mayan civilization and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the “decimal” system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, . . . , 9.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven “hundred thousands,” since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

5,000,000	$= 5 \times 1,000,000$	$= 5 \times 10^6$	Five million
+700,000	$= 7 \times 100,000$	$= 7 \times 10^5$	Seven hundred thousand
+80,000	$= 8 \times 10,000$	$= 8 \times 10^4$	Eighty thousand
+3,000	$= 3 \times 1000$	$= 3 \times 10^3$	Three thousand
+200	$= 2 \times 100$	$= 2 \times 10^2$	Two hundred
+10	$= 1 \times 10$	$= 1 \times 10^1$	Ten
+6	$= 6 \times 1$	$= 6 \times 10^0$	Six
5,783,216	Five million, seven hundred eighty-three thousand, two hundred sixteen		

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to “carry” over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let’s suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We’re in a different numerical world. As the base-ten system progresses from 10^0 to 10^1 , so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2 .

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5 , where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

	Base 5	This column coverts to base-ten	In Base-Ten
	3×5^4	$= 3 \times 625$	$= 1875$
+	0×5^3	$= 0 \times 125$	$= 0$
+	4×5^2	$= 4 \times 25$	$= 100$
+	1×5^1	$= 1 \times 5$	$= 5$
+	2×5^0	$= 2 \times 1$	$= 2$
		Total	1982

As you can see, the number 30412_5 is equivalent to 1,982 in base-ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

Example

Convert 6234_7 to a base 10 number.

Answer

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the 7s, 49s (7^2), etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$= 6 \times 7^3$	$= 6 \times 343$	$= 2058$
+	$= 2 \times 7^2$	$= 2 \times 49$	$= 98$
+	$= 3 \times 7$	$= 3 \times 7$	$= 21$
+	$= 4 \times 1$	$= 4 \times 1$	$= 4$
		Total	2181

Thus $6234_7 = 2181_{10}$

Try It Now

Convert 41065_7 to a base 10 number.

Answer

$$41065_7 = 9994_{10}$$

Visit this page in your course online to practice before taking the quiz.

Watch this video to see more examples of converting numbers in bases other than 10 into a base 10 number.

Watch this video online: https://youtu.be/TjvexlVV_gI

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ANOTHER METHOD FOR CONVERTING BETWEEN BASES

Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the “Try It Now” problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like “How could I simplify this process?” In general, that is one of the main things that mathematicians do: they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that’s the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let’s start with the decimal number, 4863_{10} . We will convert this number to base 10. Yeah, I know it’s already in base 10, but if you carefully follow what we’re doing, you’ll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is $10^3 = 1000$. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that's the base we're in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$

$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$

$$(\text{Parts of Thousands}) \times 10 \Rightarrow \text{Hundreds}$$

What we have now is 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$

$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$\begin{array}{rcl}
 4863 \div 1000 & = & \textcircled{4}.863 \\
 \downarrow & & \\
 0.863 \times 10 & = & \textcircled{8}.63 \\
 \downarrow & & \\
 0.63 \times 10 & = & \textcircled{6}.3 \\
 \downarrow & & \\
 0.3 \times 10 & = & \textcircled{3}.0
 \end{array}$$

Converting from Base 10 to Base b : Another method Note that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, b , we have the following steps we can follow:

Converting from Base 10 to Base b : Another method

1. Find the highest power of the base b that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base b .
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base b notation.

We will illustrate this procedure with some examples.

Example

Convert the base 10 number, 348_{10} , to base 5.

Answer

This is actually a conversion that we have done in a previous example. The powers of five are:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

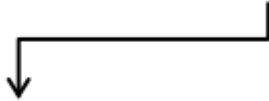
$$5^3 = 125$$

$$5^4 = 625$$

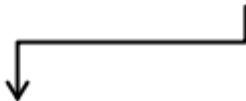
Etc...

The highest power of five that will go into 348 at least once is 5^3 .
We divide by 125 and then proceed.

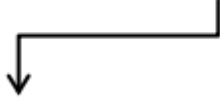
$$348 \div 5^3 = \textcircled{2}.784$$



$$0.784 \times 5 = \textcircled{3}.92$$



$$0.92 \times 5 = \textcircled{4}0.6$$



$$0.6 \times 5 = \textcircled{3}0$$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$.

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example

Convert the base 10 number, 3007_{10} , to base 5.

Answer

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

$$3007 \div 625 = \textcircled{4}.8112$$

$$0.8112 \times 5 = \textcircled{4}.056$$

$$0.056 \times 5 = \textcircled{0}.28$$

$$0.28 \times 5 = \textcircled{1}0.4$$

$$0.4 \times 5 = \textcircled{2}0.0$$

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2 s in this number.

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example

Convert the base 10 number, 63201_{10} , to base 7.

Answer

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

etc...

The highest power of 7 that will divide at least once into 63201 is 7^5 . When we do the initial division on a calculator, we get the following:

$$63201 \div 7^5 = 3.760397453$$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. **Do not round off!** Subtraction and then multiplication by seven gives:

$$63201 \div 7^5 = \textcircled{3}.760397453$$

$$0.760397453 \times 7 = \textcircled{5}.322782174$$

$$0.322782174 \times 7 = \textcircled{2}.259475219$$

$$0.259475219 \times 7 = \textcircled{1}.816326531$$

$$0.816326531 \times 7 = \textcircled{5}.714285714$$

$$0.714285714 \times 7 = \textcircled{5}.000000000$$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$.

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5 , then you expect to have 6 digits in the final answer, corresponding to the places for 7^5 , 7^4 , and so on down to 7^0 . If you find yourself with more than 6 digits due to rounding errors, you know something went wrong.

Try It Now

Convert the base 10 number, 9352_{10} , to base 5.

Answer

$$9352_{10} = 244402_5$$

Convert the base 10 number, 1500, to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

Answer

$$1500_{10} = 2001120_3$$

Visit this page in your course online to practice before taking the quiz.

The following video shows how to use a calculator to convert numbers in base 10 into other bases.

Watch this video online: <https://youtu.be/YNPTYelCels>

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THE MAYAN NUMERAL SYSTEM

Learning Objectives

- Determine the value of a number given in Mayan vertical form
- Convert between Hindu-Arabic and Mayan base 20 form
- Add Mayan numerals

Background

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base-two system, counting as follows: “one, two, two and one, two two’s, much.” Some Modern South American Tribes have a base-five system counting in this way: “one, two, three, four, hand, hand and one, hand and two,” and so on. The Babylonians used a base-

sixty (sexigesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.



1. [Chichen Itza](#)
2. [Uxmal](#)
3. [Tulum](#)
4. [Palenque](#)
5. [Bonampak, Yaxchilan](#)
6. [Tikal](#)
7. [Altun Ha](#)
8. [Copán](#)

The Mayan civilization is generally dated from 1500 BCE to 1700 CE. The Yucatan Peninsula (see figure 16 (Note: http://www.gorp.com/gorp/location/latamer/map_maya.htm)) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a philosophy with time as divine and eternal. (Note: Bidwell, James; Mayan Arithmetic in *Mathematics Teacher*, Issue 74 (Nov., 1967), p. 762–68.) The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.

There were two numeral systems developed by the Mayans—one for the common people and one for the priests. Not only did these two systems use different symbols, they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads, (Note: <http://www.ukans.edu/~lctls/Mayan/numbers.html>) like the sample to the left (Note: <http://www.ukans.edu/~lctls/Mayan/numbers.html>) Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.



Powers	Base-Ten Value	Place Name
20^7	12,800,000,000	Hablat
20^6	64,000,000	Alau

20^5	3,200,000	Kinchil
20^4	160,000	Cabal
20^3	8,000	Pic
20^2	400	<i>Bak</i>
20^1	20	Kal
20^0	1	Hun


The Mayan Number System

Instead, we will focus on the numeration system of the “common” people, which used a more consistent base system. As we stated earlier, the Mayans used a base-20 system, called the “vigesimal” system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table you can see the place value in its vertical format. (Note: Bidwell)

In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right. (Note: http://www.vpds.wsu.edu/fair_95/gym/UM001.html)

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the **bottom** of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place...four bars would be converted to one dot in the next place up. It's the same as 10 getting converted to a 1 in the next place up when we carry during addition.

Number	Vertical Form	Number	Vertical Form
0		10	<u>==</u>
1	o	11	<u>o</u>
2	o o	12	<u>o o</u>
3	o o o	13	<u>o o o</u>
4	o o o o	14	<u>o o o o</u>
5	—	15	<u>==</u>
6	<u>o</u>	16	<u>o</u>
7	<u>o o</u>	17	<u>o o</u>
8	<u>o o o</u>	18	<u>o o o</u>
9	<u>o o o o</u>	19	<u>o o o o</u>

Example

What is the value of this number, which is shown in vertical form?

• • •

• • •
=====

Answer

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.

• • • ← 20's

• • •
===== ← 1's

Hence we can write this number in base-ten as:

$$(3 \times 20^1) + (13 \times 20^0) = (3 \times 20^1) + (13 \times 1) = 60 + 13 = 73$$

Example

What is the value of the following Mayan number?

• • •
=====

⬢

•
=====

Answer

This number has 11 in the ones place, zero in the 20s place, and 18 in the $20^2 = 400$ s place. Hence, the value of this number in base-ten is:

$$18 \times 400 + 0 \times 20 + 11 \times 1 = 7211.$$

Try It Now

Convert the Mayan number below to base 10.

• • •

• • •
=====

• •

Answer

1562

Visit this page in your course online to practice before taking the quiz.

Example

Convert the base 10 number 3575_{10} to Mayan numerals.

Answer

This problem is done in two stages. First we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$$3575 \div 400 = 8.9375$$

$$0.9375 \times 20 = 18.75$$

$$0.75 \times 20 = 15.0$$

This means that $3575_{10} = 8,18,15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20s place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400s place, so that's one bar and three dots on the top. We get the following:



Note that in the previous example a new notation was used when we wrote $8,18,15_{20}$. The commas between the three numbers 8, 18, and 15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing numbers with bases bigger than 10

When the base of a number is larger than 10, separate each "digit" with a comma to make the separation of digits clear.

For example, in base 20, to write the number corresponding to $17 \times 20^2 + 6 \times 20^1 + 13 \times 20^0$, we'd write $17,6,13_{20}$.

Try It Now

Convert the base 10 number 10553_{10} to Mayan numerals.

Answer

$$10553_{10} = 1, 6, 7, 13_{20}$$



Visit this page in your course online to practice before taking the quiz.
Convert the base 10 number 5617_{10} to Mayan numerals.

Answer

$5617_{10} = 14, 0, 17_{20}$. Note that there is a zero in the 20's place, so you'll need to use the appropriate zero symbol in between the ones and 400's places.



In the following video we present more examples of how to write numbers using Mayan numerals as well as converting numerals written in Mayan for into base 10 form.

Watch this video online: <https://youtu.be/gPUOrcilVS0>

The next video shows more examples of converting base 10 numbers into Mayan numerals.

Watch this video online: https://youtu.be/LrHNXoqQ_I

Adding Mayan Numbers

When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

Example

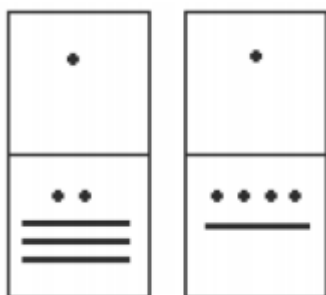
Add, in Mayan, the numbers 37 and 29:



(Note: <http://forum.swarthmore.edu/k12/mayan.math/mayan2.html>)

Answer

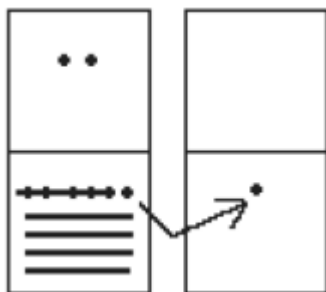
First draw a box around each of the vertical places. This will help keep the place values from being mixed up.



Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:

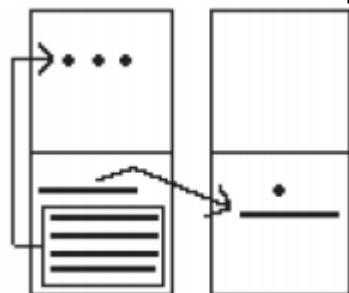


You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

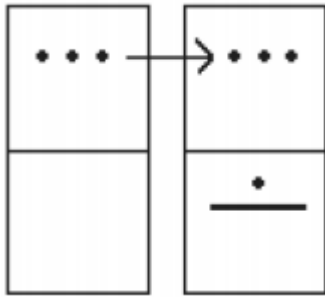


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place.*

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



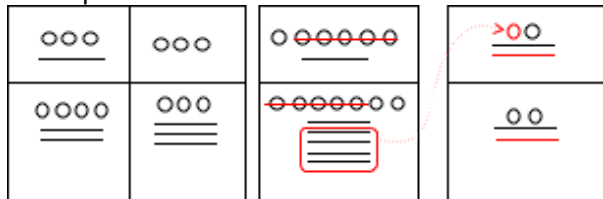
We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that $37 + 29 = 66$, so we have done this addition correctly. Is it easier to just do it in base-ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

Try It Now

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Answer

A sample solution is shown.



In the last video we show more examples of adding Mayan numerals.

Watch this video online: <https://youtu.be/NpH5oMCrubM>

In this module, we have briefly sketched the development of numbers and our counting system, with the emphasis on the “brief” part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.



We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. What is important to note is that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

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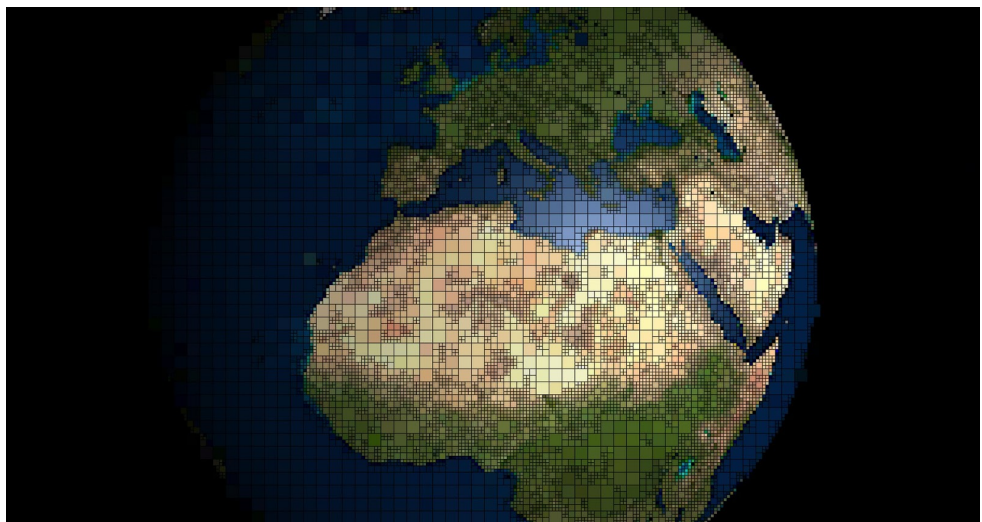
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PUTTING IT TOGETHER: HISTORICAL COUNTING SYSTEMS

Computers speak **binary**.

The binary, or base 2, positional number system uses only two digits, 0 and 1, which makes it ideal for computers whose basic components typically exist in two states, *off* (0) or *on* (1).

However, binary numbers can be very tedious for human readers to interpret. For example, can you figure out what decimal number is represented by the following binary number?



11011110110000010101100100011111

There are 33 digits in this numeral! We may estimate roughly how big this number is by working out just the first few digits (we'll work out just four digits, but you could go further if you wanted to).

$$\begin{aligned}
 &1 \times 2^{32} + 1 \times 2^{31} + 0 \times 2^{30} + 1 \times 2^{29} \\
 &= 4,294,967,296 + 2,147,483,648 + 0 + 536,870,912 \\
 &\approx 7,000,000,000
 \end{aligned}$$

This number, about 7 billion, represents the world population on January 1st, 2017 according to the website, Population.City.

Now if we wanted to work out the decimal value of this number down to the last digit, there is a nice shortcut. First let's separate the digits into groups of four.

1 1011 1101 1000 0010 1011 0010 0011 1111

Each group may now be thought of as a numeral in base $2^4 = 16$. The base 16, or hexadecimal, number system requires 16 digits. In particular, the digits corresponding to decimal 10 through 15 are usually written as the letters A through F, as indicated in Table I.

Table I: Hexadecimal digits and their equivalent decimal representations.

Hexa- decimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Furthermore, there is a one-to-one correspondence between every possible 4-digit binary expression and the hexadecimal digits, as shown in Table II. It is this correspondence that makes hexadecimal a natural medium for writing large binary numbers in a more human-readable way.

Table II: Binary-hexadecimal conversion chart.

Binary	Hex.	Binary	Hex.	Binary	Hex.	Binary	Hex.
0000	0	0100	4	1000	8	1100	C
0001	1	0101	5	1001	9	1101	D
0010	2	0110	6	1010	A	1110	E
0011	3	0111	7	1011	B	1111	F

So let's see what the population of the Earth looks like in hexadecimal notation. Note that the leading "1" in our numeral should be interpreted as "0001."

0001	1011	1101	1000	0010	1011	0010	0011	1111
1	B	D	8	2	B	2	3	F

The resulting hexadecimal number, 1BD82B23F, is much easier to write down and to work with than the original binary representation, at least for us non-computers.

Finally, let's figure out the exact decimal representation of the number using what we have learned in this module about place-value systems.

Digit	Place value	Computed values
1	16^8	$1 \times 16^8 = 4,294,967,296$
B	16^7	$11 \times 16^7 = 2,952,790,016$
D	16^6	$13 \times 16^6 = 218,103,808$
8	16^5	$8 \times 16^5 = 8,388,608$
2	16^4	$2 \times 16^4 = 131,072$
B	16^3	$11 \times 16^3 = 45,056$
2	16^2	$2 \times 16^2 = 512$
3	16	$3 \times 16 = 48$
F	1	$15 \times 1 = 15$
	Total:	7,474,426,431

Almost seven and a half billion people call this planet home. That's a lot of people regardless whether the amount is expressed in decimal, hexadecimal, or binary notation.

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GENERAL PROBLEM SOLVING

WHY IT MATTERS: GENERAL PROBLEM SOLVING

Critical Thinking

Thinking comes naturally. You don't have to make it happen—it just does. But you can make it happen in different ways. For example, you can think positively or negatively. You can think with “heart” and you can think with rational judgment. You can also think strategically and analytically, and mathematically and scientifically. These are a few of multiple ways in which the mind can process thought.

What are some forms of thinking you use? When do you use them, and why?

As a college student, you are tasked with engaging and expanding your thinking skills. One of the most important of these skills is critical thinking. Critical thinking is important because it relates to nearly all tasks, situations, topics, careers, environments, challenges, and opportunities. It's a “domain-general” thinking skill—not a thinking skill that's reserved for a one subject alone or restricted to a particular subject area.

Great leaders have highly attuned critical thinking skills, and you can, too. In fact, you probably have a lot of these skills already. Of all your thinking skills, critical thinking may have the greatest value.

What Is Critical Thinking?

Critical thinking is clear, reasonable, reflective thinking focused on deciding what to believe or do. It means asking probing questions like, “How do we know?” or “Is this true in every case or just in this instance?” It involves being skeptical and challenging assumptions, rather than simply memorizing facts or blindly accepting what you hear or read.

Who are critical thinkers, and what characteristics do they have in common? Critical thinkers are usually curious and reflective people. They like to explore and probe new areas and seek knowledge, clarification, and new solutions. They ask pertinent questions, evaluate statements and arguments, and they distinguish between facts and opinion. They are also willing to examine their own beliefs, possessing a manner of humility that allows them to admit lack of knowledge or understanding when needed. They are open to changing their mind. Perhaps most of all, they actively enjoy learning, and seeking new knowledge is a lifelong pursuit.

This may well be you!

Critical Thinking IS	Critical Thinking is NOT
Skepticism	Memorizing
Examining assumptions	Group thinking
Challenging reasoning	Blind acceptance of authority

Critical Thinking IS	Critical Thinking is NOT
Uncovering biases	

The following video, from Lawrence Bland, presents the major concepts and benefits of critical thinking.

Watch this video online: <https://youtu.be/WiSkIIUblo>

Critical Thinking and Logic

Critical thinking is fundamentally a process of questioning information and data. You may question the information you read in a textbook, or you may question what a politician or a professor or a classmate says. You can also question a commonly-held belief or a new idea. With critical thinking, anything and everything is subject to question and examination for the purpose of logically constructing reasoned perspectives.

Questions of Logic in Critical Thinking

Let's use a simple example of applying logic to a critical-thinking situation. In this hypothetical scenario, a man has a PhD in political science, and he works as a professor at a local college. His wife works at the college, too. They have three young children in the local school system, and their family is well known in the community. The man is now running for political office. Are his credentials and experience sufficient for entering public office? Will he be effective in the political office? Some voters might believe that his personal life and current job, on the surface, suggest he will do well in the position, and they will vote for him. In truth, the characteristics described don't guarantee that the man will do a good job. The information is somewhat irrelevant. What else might you want to know? How about whether the man had already held a political office and done a good job? In this case, we want to ask, How much information is adequate in order to make a decision based on logic instead of assumptions?

The following questions are ones you may apply to formulating a logical, reasoned perspective in the above scenario or any other situation:

1. *What's happening?* Gather the basic information and begin to think of questions.
2. *Why is it important?* Ask yourself why it's significant and whether or not you agree.
3. *What don't I see?* Is there anything important missing?
4. *How do I know?* Ask yourself where the information came from and how it was constructed.
5. *Who is saying it?* What's the position of the speaker and what is influencing them?
6. *What else? What if?* What other ideas exist and are there other possibilities?

Problem-Solving with Critical Thinking

For most people, a typical day is filled with critical thinking and problem-solving challenges. In fact, critical thinking and problem-solving go hand-in-hand. They both refer to using knowledge, facts, and data to solve problems effectively. But with problem-solving, you are specifically identifying, selecting, and defending your solution.

Problem-Solving Action Checklist

Problem-solving can be an efficient and rewarding process, especially if you are organized and mindful of critical steps and strategies. Remember, too, to assume the attributes of a good critical thinker. If you are curious, reflective, knowledge-seeking, open to change, probing, organized, and ethical, your challenge or problem will be less of a hurdle, and you'll be in a good position to find intelligent solutions.

	STRATEGIES	ACTION CHECKLIST (Note: "Student Success-Thinking Critically In Class and Online." <i>Critical Thinking Gateway</i>. St Petersburg College, n.d. Web. 16 Feb 2016.)
1	Define the problem	<ul style="list-style-type: none"> • Identify the problem • Provide as many supporting details as possible • Provide examples • Organize the information logically
2	Identify available solutions	<ul style="list-style-type: none"> • Use logic to identify your most important goals • Identify implications and consequences • Identify facts • Compare and contrast possible solutions
3	Select your solution	<ul style="list-style-type: none"> • Use gathered facts and relevant evidence • Support and defend solutions considered valid • Defend your solution

Critical Thinking, Problem Solving, and Math

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won't need) to answer the question.

In this section, we will review several basic but powerful algebraic ideas: **percents**, **rates**, and **proportions**. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don't have perfect information.

Learning Objectives

Solve problems involving percents, proportions, and rates.

- Describing quantities and how they change
- Write an equivalent fraction or decimal given a percent
- Find a percent of a whole
- Calculate absolute and relative change given two quantities
- Express a relationship as a rate
- Write a proportion equation given two rates or ratios, solve the proportion equation
- Determine when two quantities don't scale proportionally, or more information is needed to determine whether they do

Solve problems using basic geometry

- Area
- Volume
- Proportions, similar triangles, ratios applied to geometric problems

Use mathematical problem solving and estimation techniques.

- Define and implement a “solution pathway” for solving mathematical problems
- Calculate sales tax, property tax
- Calculate flat tax, progressive tax, and regressive tax

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INTRODUCTION: PROPORTIONAL RELATIONSHIPS AND A BIT OF GEOMETRY

Learning Objectives

In this lesson you will learn how to do the following:

- Given the part and the whole, write a percent
- Calculate both relative and absolute change of a quantity
- Calculate tax on a purchase

In the 2004 vice-presidential debates, Democratic contender John Edwards claimed that US forces have suffered “90% of the coalition casualties” in Iraq. Incumbent Vice President Dick Cheney disputed this, saying that in fact Iraqi security forces and coalition allies “have taken almost 50 percent” of the casualties. (Note: http://www.factcheck.org/cheney_edwards_mangle_facts.html)

Who was correct? How can we make sense of these numbers?

In this section, we will show how the idea of percent is used to describe parts of a whole. Percents are prevalent in the media we consume regularly, making it imperative that you understand what they mean and where they come from.

We will also show you how to compare different quantities using proportions. Proportions can help us understand how things change or relate to each other.

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PERCENTS

In the 2004 vice-presidential debates, Democratic contender John Edwards claimed that US forces have suffered “90% of the coalition casualties” in Iraq. Incumbent Vice President Dick Cheney disputed this, saying that in fact Iraqi security forces and coalition allies “have taken almost 50 percent” of the casualties. (Note: http://www.factcheck.org/cheney_edwards_mangle_facts.html)

Who was correct? How can we make sense of these numbers?

Percent literally means “per 100,” or “parts per hundred.” When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200} = \frac{10}{25} = \frac{40}{100}$.



A visual depiction of 40%

Percent

If we have a *part* that is some *percent* of a *whole*, then $\text{percent} = \frac{\text{part}}{\text{whole}}$, or equivalently, $\text{part} \cdot \text{whole} = \text{percent}$.

To do the calculations, we write the percent as a decimal.

For a refresher on basic percentage rules, using the examples on this page, view the following video.

Watch this video online: <https://youtu.be/Z229RysttR8>

Examples

In a survey, 243 out of 400 people state that they like dogs. What percent is this?

Answer

$$\frac{243}{400} = 0.6075 = \frac{60.75}{100} \text{ This is } 60.75\%.$$

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example

Write each as a percent:

1. $\frac{1}{4}$
2. 0.02
3. 2.35

Answer

1. $\frac{1}{4} = 0.25 = 25\%$
2. $0.02 = 2\%$
3. $2.35 = 235\%$

Try It Now

Throughout this text, you will be given opportunities to answer questions and know immediately whether you answered correctly. To answer the question below, do the calculation on a separate piece of paper and enter your answer in the box. Click on the submit button , and if you are correct, a green box will appear around your answer. If you are incorrect, a red box will appear. You can click on “Try Another Version of This Question” as many times as you like. Practice all you want!
Visit this page in your course online to practice before taking the quiz.

Example

In the news, you hear “tuition is expected to increase by 7% next year.” If tuition this year was \$1200 per quarter, what will it be next year?

Answer

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year’s tuition: $\$1200(1.07) = \1284 .

Alternatively, we could have first calculated 7% of \$1200: $\$1200(0.07) = \84 .

Notice this is not the expected tuition for next year (we could only wish). Instead, this is the expected increase, so to calculate the expected tuition, we’ll need to add this change to the previous year’s tuition: $\$1200 + \$84 = \$1284$.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

The value of a car dropped from \$7400 to \$6800 over the last year. What percent decrease is this?

Answer

To compute the percent change, we first need to find the dollar value change: $\$6800 - \$7400 = -\$600$. Often we will take the absolute value of this amount, which is called the **absolute change**: $|-600| = 600$. Since we are computing the decrease relative to the starting value, we compute this percent out of \$7400:

$$\frac{600}{7400} = 0.081 = 8.1\% \text{ decrease. This is called a } \mathbf{relative \ change}.$$

Absolute and Relative Change

Given two quantities,

Absolute change = $|\text{ending quantity} - \text{starting quantity}|$

Relative change: $\frac{\text{absolute change}}{\text{starting quantity}}$

- Absolute change has the same units as the original quantity.
- Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

For a deeper dive on absolute and relative change, using the examples on this page, view the following video.

Watch this video online: <https://youtu.be/QjVeurkg8CQ>

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a “gateway” drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn’t true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs. (Note:

<http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics>)

Example

There are about 75 QFC supermarkets in the United States. Albertsons has about 215 stores. Compare the size of the two companies.

Answer

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215 - 75 = 140$. From this, we could say “Albertsons has 140 more stores than QFC.” However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75} = 1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215} = 0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons

relative to QFC: $\frac{215}{75} = 2.867$, which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could

calculate the size of QFC relative to Albertsons: $\frac{75}{215} = 0.349$, which tells us that QFC is 34.9% of the size of Albertsons.

Example

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Answer

To answer this question, suppose the value started at \$100. After one week, the value dropped by 60%: $\$100 - \$100(0.60) = \$100 - \$60 = \$40$.

In the next week, notice that base of the percent has changed to the new value, \$40. Computing the 75% increase: $\$40 + \$40(0.75) = \$40 + \$30 = \$70$.

In the end, the stock is still \$30 lower, or $\frac{\$30}{100} = 30\%$ lower, valued than it started.

A video walk-through of this example can be seen here.

Watch this video online: <https://youtu.be/4HNxwYMTNl8>

Consideration of the base of percentages is explored in this video, using the examples on this page.

Watch this video online: <https://youtu.be/nygw69JqwoQ>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

A *Seattle Times* article on high school graduation rates reported “The number of schools graduating 60 percent or fewer students in four years—sometimes referred to as ‘dropout factories’—decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half.”

1. Is the “decreased by 17” number a useful comparison?
2. Considering the last sentence, can we conclude that the number of “dropout factories” was originally 34?

Answer

1. This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of “dropout factories” dropped from 20 to 3, that’d be a very significant change, but if the number dropped from 217 to 200, that’d be less of an improvement.
2. The last sentence provides relative change, which helps put the first sentence in perspective. We can estimate that the number of “dropout factories” was probably previously around 34. However, it’s possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example

Let’s return to the example at the top of this page. In the 2004 vice-presidential debates, Democratic candidate John Edwards claimed that US forces have suffered “90% of the coalition casualties” in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies “have taken almost 50 percent” of the casualties. Who is correct?

Answer

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward’s claim was a percent with coalition forces as the base of the percent, while Cheney’s claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

A detailed explanation of these examples can be viewed here.

Watch this video online: <https://youtu.be/Svlu2Lurmsc>

Think About It

In the 2012 presidential elections, one candidate argued that “the president’s plan will cut \$716 billion from Medicare, leading to fewer services for seniors,” while the other candidate rebuts that “our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures.” Are these claims in conflict, in agreement, or not comparable because they’re talking about different things?



We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

Example

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

Answer

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

Example

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

Answer

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then that player made $200(0.40) = 80$ 2-point shots and $100(0.30) = 30$ 3-point shots. Overall, they player made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

For more information about these cautionary tales using percentages, view the following.

Watch this video online: <https://youtu.be/vtgEkQUB5F8>

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PROPORTIONS AND RATES

If you wanted to power the city of Lincoln, Nebraska using wind power, how many wind turbines would you need to install? Questions like these can be answered using rates and proportions.



Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Answer

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{ gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios. For an overview on rates and proportions, using the examples on this page, view the following video. Watch this video online: <https://youtu.be/aZrio6ztHKE>

Example

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x .

Answer

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

Example

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Answer

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x , to represent the unknown quantity—the mile distance between the cities.

$\frac{\frac{1}{2} \text{ map inch}}{3 \text{ miles}} = \frac{2\frac{1}{4} \text{ map inches}}{x \text{ miles}}$	Multiply both sides by x and rewriting the mixed number
--	---

$$\frac{1}{3} \cdot x = \frac{9}{4}$$

Multiply both sides by 3

$$\frac{1}{2}x = \frac{27}{4}$$

Multiply both sides by 2 (or divide by $\frac{1}{2}$)

$$x = \frac{27}{2} = 13\frac{1}{2} \text{ miles}$$

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Answer

We could certainly answer this question using a proportion: $\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit “cancels” and we’re left with a number of miles:

$$40 \text{ gallons} \cdot \frac{20 \text{ miles}}{1 \text{ gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallons}} = 800 \text{ miles}$$

Notice if instead we were asked “how many gallons are needed to drive 50 miles?” we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we’re left with gallons:

$$50 \text{ miles} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

A worked example of this last question can be found in the following video.

Watch this video online: <https://youtu.be/jYwi3YqP0Wk>

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

You have likely encountered distance, rate, and time problems in the past. This is likely because they are easy to visualize and most of us have experienced them first hand. In our next example, we will solve distance, rate and time problems that will require us to change the units that the distance or time is measured in.

Example

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Answer

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don’t, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour}$$

Now we can multiply by the 15 miles/hr

$$\frac{1}{180} \text{ hour} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} = \frac{1}{12} \text{ mile}$$

Now we can convert to feet

$$\frac{1}{12} \text{ mile} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{15 \text{ miles}}{1 \text{ miles}} = \frac{5280 \text{ feet}}{1 \text{ mile}} = \frac{1}{180} \text{ hour}$$

View the following video to see this problem worked through.

Watch this video online: <https://youtu.be/fyOclClVipM>

Try It Now

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Visit this page in your course online to practice before taking the quiz.

Example

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Answer

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100\text{ft}^2} = \frac{n \text{ tiles}}{400\text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example

Suppose a small company spends \$1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend \$10,000?

Answer

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely

only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Matters of scale in this example and the previous one are explained in more detail here.

Watch this video online: <https://youtu.be/-e2typcrhLE>

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Examples

The 2010 U.S. military budget was \$683.7 billion. To gain perspective on how much money this is, answer the following questions.

1. What would the salary of each of the 1.4 million Walmart employees in the US be if the military budget were distributed evenly amongst them?
2. If you distributed the military budget of 2010 evenly amongst the 300 million people who live in the US, how much money would you give to each person?
3. If you converted the US budget into \$100 bills, how long would it take you to count it out – assume it takes one second to count one \$100 bill.

Answer

Here we have a very large number, about \$683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

1. If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over \$488,000.
2. There are about 300 million people in the U.S. The military budget is about \$2,200 per person.
3. If you were to put \$683.7 billion in \$100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example

Compare the electricity consumption per capita in China to the rate in Japan.

Answer

To address this question, we will first need data. From the CIA (Note:

<https://www.cia.gov/library/publications/the-world-factbook/rankorder/2042rank.html>) website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank, (Note: <http://data.worldbank.org/indicator/SP.POP.TOTL>) we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Computing the consumption per capita for each country:

$$\text{China: } \frac{4,693,000,000,000 \text{ KWH}}{1,344,130,000 \text{ people}} \approx 3491.5 \text{ KWH per person}$$

Japan: $\frac{859,700,000,000 \text{ KWH}}{127,817,277 \text{ people}} \approx 6726 \text{ KWH per person}$

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

Working with large numbers is examined in more detail in this video.

Watch this video online: <https://youtu.be/rCLh8ZvSQr8>

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A BIT OF GEOMETRY

Geometric shapes, as well as area and volumes, can often be important in problem solving.



Let's start things off with an example, rather than trying to explain geometric concepts to you.

Example

You are curious how tall a tree is, but don't have any way to climb it. Describe a method for determining the height.

Answer

There are several approaches we could take. We'll use one based on triangles, which requires that it's a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall, and cast a 1.5 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we'll denote by h :

$$\frac{6\text{ft. tall}}{1.5\text{ft. shadow}} = \frac{h\text{ft. tall}}{15\text{ft. shadow}}$$

Multiplying both sides by 15, we get $h = 60$. The tree is about 60 ft tall.

Similar Triangles

We introduced the idea of similar triangles in the previous example. One property of geometric shapes that we have learned is a helpful problem-solving tool is that of similarity. If two triangles are the same, meaning the angles between the sides are all the same, we can find an unknown length or height as in the last example. This idea of similarity holds for other geometric shapes as well.

Guided Example

Mary was out in the yard one day and had her two daughters with her. She was doing some renovations and wanted to know how tall the house was. She noticed a shadow 3 feet long when her daughter was standing 12 feet from the house and used it to set up figure 1.

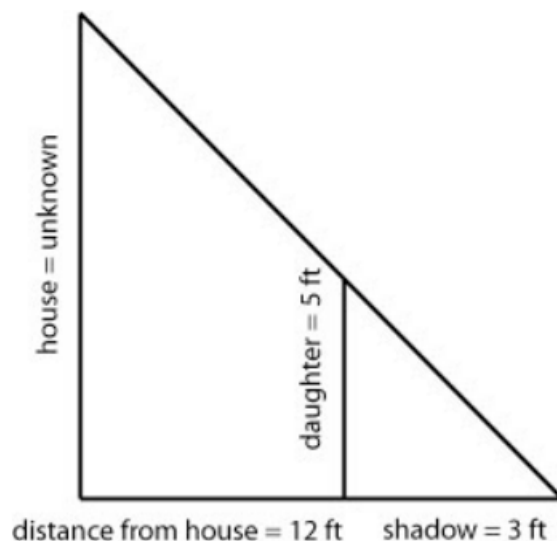


Figure 1.

We can take that drawing and separate the two triangles as follows allowing us to focus on the numbers and the shapes.

These triangles are what are called **similar triangles**. They have the same angles and sides in *proportion* to each other. We can use that information to determine the height of the house as seen in figure 2.

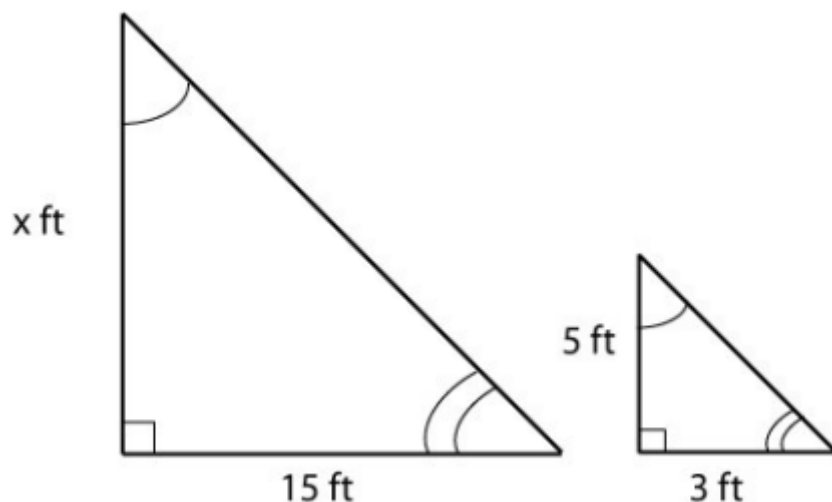


Figure 2.

To determine the height of the house, we set up the following proportion:

$$\frac{x}{15} = \frac{5}{3}$$

Then, we solve for the unknown x by using cross products as we have done before:

$$x = \frac{5 \times 15}{3} = \frac{75}{3} = 25$$

Therefore, we can conclude that the house is 25 feet high.

Try It Now

Visit this page in your course online to practice before taking the quiz.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes:

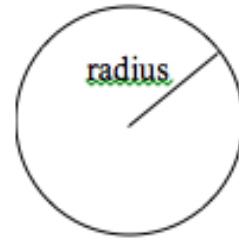
Areas

Rectangle	Circle, radius r
Area: $L \times W$	Area: πr^2
Perimeter: $2l + 2W$	Circumference $2\pi r$

W



L

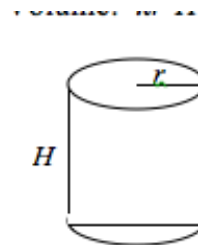
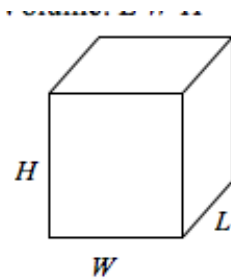


Rectangular Box

Volume: $L \times W \times H$

Cylinder

Volume: $\pi r^2 h$



In our next two examples, we will combine the ideas we have explored about ratios with the geometry of some basic shapes to answer questions. In the first example, we will predict how much dough will be needed for a pizza that is 16 inches in diameter given that we know how much dough it takes for a pizza with a diameter of 12 inches. The second example uses the volume of a cylinder to determine the number of calories in a marshmallow.

Examples

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

Answer

To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle, $A = \pi r^2$:

A 12" pizza has radius 6 inches, so the area will be $\pi 6^2 =$ about 113 square inches.

A 16" pizza has radius 8 inches, so the area will be $\pi 8^2 =$ about 201 square inches.

Notice that if both pizzas were 1 inch thick, the volumes would be 113 in³ and 201 in³ respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas,

we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16" pizza:

$$\frac{10 \text{ ounces}}{113\text{in}^2} = \frac{x \text{ ounces}}{201\text{in}^2}$$

Multiply both sides by 201

$$x = 201 \cdot \frac{10}{113} = \text{about } 17.8 \text{ ounces of dough for a } 16'' \text{ pizza.}$$

It is interesting to note that while the diameter is $\frac{16}{12} = 1.33$ times larger, the dough required, which scales with area, is $1.33^2 = 1.78$ times larger.

The following video illustrates how to solve this problem.

Watch this video online: <https://youtu.be/e75bk1qCsUE>

Example

A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

Answer

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7 \text{ units}^3$.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8 \text{ units}^3$.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have:

$$118.8 \text{ units}^3 \cdot \frac{25 \text{ calories}}{33.7 \text{ units}^3} = 88.1 \text{ calories}$$

It is interesting to note that while the diameter and height are about 1.5 times larger for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times larger.

For more about the marshmallow example, watch this video.

Watch this video online: <https://youtu.be/QJgGpxzRt6Y>

Try It Now

A website says that you'll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

Answer

The original sandbox has volume 64ft^3 . The smaller sandbox has volume 24ft^3 .

$$\frac{48\text{bags}}{64\text{ft}^3} = \frac{x \text{ bags}}{24\text{ft}^3} \text{ results in } x = 18 \text{ bags.}$$

Mary (from the application that started this topic), decides to use what she knows about the height of the roof to measure the height of her second daughter. If her second daughter casts a shadow that is 1.5 feet

long when she is 13.5 feet from the house, what is the height of the second daughter? Draw an accurate diagram and use similar triangles to solve.

Answer

2.5 ft

Visit this page in your course online to practice before taking the quiz.

In the next section, we will explore the process of combining different types of information to answer questions.

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INTRODUCTION: SOLVING PROBLEMS WITH MATH

Learning Objectives

- Identify and apply a solution pathway for multi-step problems

In this section we will bring together the mathematical tools we've reviewed, and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does **not** work well with real life problems, however. Read on to learn how to use a generalized problem solving approach to solve a wide variety of quantitative problems, including how taxes are calculated.

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PROBLEM SOLVING AND ESTIMATING

Problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very

few interesting questions can be answered in one mathematical step; often times you will need to chain together a **solution pathway**, a series of steps that will allow you to answer the question.

Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

In the first example, we will need to think about time scales, we are asked to find how many times a heart beats in a year, but usually we measure heart rate in beats per minute.

Examples

How many times does your heart beat in a year?

Answer

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this to beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

The technique that helped us solve the last problem was to get the number of heartbeats in a minute translated into the number of heartbeats in a year. Converting units from one to another, like minutes to years is a common tool for solving problems.

In the next example, we show how to infer the thickness of something too small to measure with every-day tools. Before precision instruments were widely available, scientists and engineers had to get creative with ways to measure either very small or very large things. Imagine how early astronomers inferred the distance to stars, or the circumference of the earth.

Example

How thick is a single sheet of paper? How much does it weigh?

Answer

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or } = 0.16 \text{ ounces per sheet.}$$

The first two example questions in this set are examined in more detail here.

Watch this video online: <https://youtu.be/xF5BNEr0gjo>

We can infer a measurement by using scaling. If 500 sheets of paper is two inches thick, then we could use proportional reasoning to infer the thickness of one sheet of paper.

In the next example, we use proportional reasoning to determine how many calories are in a mini muffin when you are given the amount of calories for a regular sized muffin.

Example

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Answer

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$$12 \text{ muffins} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000 \text{ calories for the whole recipe}$$

$$\frac{3000 \text{ calories}}{20 \text{ mini-muffins}} = \text{gives 150 calories per mini-muffin}$$

$$4 \text{ mini-muffins} \cdot \frac{150 \text{ calories}}{\text{mini-muffin}} = \text{totals 600 calories consumed.}$$

View the following video for more about the zucchini muffin problem.

Watch this video online: <https://youtu.be/NVCwFO-w2z4>

We have found that ratios are very helpful when we know some information but it is not in the right units, or parts to answer our question. Making comparisons mathematically often involves using ratios and proportions. For the last

Example

You need to replace the boards on your deck. About how much will the materials cost?

Answer

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384 ft^2 . From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about \$7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$$8 \text{ feet} \cdot 4 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667 \text{ ft}^2. \text{ The cost per square foot is then } \frac{\$7.50}{2.667 \text{ ft}^2} = \$2.8125 \text{ per ft}^2.$$

This will allow us to estimate the material cost for the whole 384 ft^2 deck

$$\$384 \text{ ft}^2 \cdot \frac{\$2.8125}{\text{ft}^2} = \$1080 \text{ total cost.}$$

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

This example is worked through in the following video.

Watch this video online: <https://youtu.be/adPGfTy-Pc>

Example

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Answer

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year.

From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

$$\textbf{Sonata: } 9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$$

$$\textbf{Hybrid: } 9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$$

If gas in your area averages about \$3.50 per gallon, we can use that to find the running cost:

$$\textbf{Sonata: } 460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$$

$$\textbf{Hybrid: } 332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$$

The hybrid will save \$450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{\$1612.45} = 0.279 = 27.9\%$

lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since “is it worth it” implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about \$25,850, compared to the base model for the regular Sonata, at \$20,895.

To better answer the “is it worth it” question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs \$4965 more. With gas savings of \$451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can’t make it for you.

This question pulls together all the skills discussed previously on this page, as the video demonstration illustrates.

Watch this video online: <https://youtu.be/HXmc-EkOYJE>

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TAXES



Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the environmental protection agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

Example: Sales Tax

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a \$140 purchase?

Answer

The sales tax will be 9.3% of \$140. To compute this, we multiply \$140 by the percent written as a decimal: $\$140(0.093) = \13.02 .

Visit this page in your course online to practice before taking the quiz.

When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the **effective tax rate**: the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

Example: Property Tax

Jaquim paid \$3,200 in property taxes on his house valued at \$215,000 last year. What is the effective tax rate?

Answer

We can compute the equivalent percentage: $3200/215000 = 0.01488$, or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

- A **flat tax**, or proportional tax, charges a constant percentage rate.
- A **progressive tax** increases the percent rate as the base amount increases.
- A **regressive tax** decreases the percent rate as the base amount increases.

Example: Federal Income Tax

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

For a single person in 2011, adjusted gross income (income after deductions) under \$8,500 was taxed at 10%. Income over \$8,500 but under \$34,500 was taxed at 15%.

Earning \$10,000

Stephen earned \$10,000 in 2011. He would pay 10% on the portion of his income under \$8,500, and 15% on the income over \$8,500.

$$8500(0.10) = 850 \quad 10\% \text{ of } \$8500$$

$$1500(0.15) = 225 \quad 15\% \text{ of the remaining } \$1500 \text{ of income}$$

$$\text{Total tax:} = \$1075$$

What was Stephen's effective tax rate?

Answer

The effective tax rate paid is $1075/10000 = 10.75\%$

Earning \$30,000

D'Andrea earned \$30,000 in 2011. She would also pay 10% on the portion of her income under \$8,500, and 15% on the income over \$8,500.

$$8500(0.10) = 850 \quad 10\% \text{ of } \$8500$$

$$21500(0.15) = 3225 \quad 15\% \text{ of the remaining } \$21500 \text{ of income}$$

$$\text{Total tax:} = \$4075$$

What was D'Andrea's effective tax rate?

Answer

The effective tax rate paid is $4075/30000 = 13.58\%$.

Notice that the effective rate has increased with income, showing this is a progressive tax.

Example: Gasoline Tax

A gasoline tax is a flat tax when considered in terms of consumption. A tax of, say, \$0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at \$4 a gallon would pay \$3 in tax, which is $\$3/\$40 = 7.5\%$. Someone buying 30 gallons of gas at \$4 a gallon would pay \$9 in tax, which is $\$9/\$120 = 7.5\%$, the same effective rate.

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning \$30,000 a year and someone earning \$60,000 a year will drive about the same amount. If both pay \$60 in gasoline taxes over a year, the person earning \$30,000 has paid 0.2% of their income, while the person earning \$60,000 has paid 0.1% of their income in gas taxes.

Try It Now

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

Answer

While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

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PUTTING IT TOGETHER: GENERAL PROBLEM SOLVING

Now that you have seen a number of general problem solving techniques and plenty of examples in this module, let's try to put it all together. In real life, sometimes we have to make big decisions. A major career change or relocation can throw your life into disarray, but perhaps even more importantly, such a change can have lasting effects on your financial future.



Suppose you are currently working as a bank teller in your hometown. Your job pays \$11 per hour, with 10% taxes taken out of each paycheck. For simplicity, assume you have to work 260 days of the year (you don't work weekends), 8 hours a day. By carefully budgeting you can keep your monthly expenses to about \$1500.

But now another job offer has come up. You have the opportunity to become an assistant manager. The only catch is that you'll have to move to a new branch of the bank opening up in a nearby metropolitan area. So here are the details:

The assistant manager salary starts at \$32,000. This puts you into the next tax bracket, which means that you will pay \$2500 plus 15% of the amount earned over \$25,000. Because you are moving to a bigger city, your living expenses will rise as well. After a little research, you determine that your monthly expenses will probably be around \$2000.

Should you make the move and take the job offer?

This is a tough decision! We should do some calculations first. How much money do you make in a full year as a bank teller? First let's find out how many total hours you work in a year.

$$8 \text{ hrs./day} \times 260 \text{ days} = 2080 \text{ hrs.}$$

Next, multiply by the hourly wage to determine your **gross income**.

$$2080 \text{ hrs.} \times 11 \text{ \$/hr.} = \$22,880$$

But don't forget about taxes! You will have to pay 10% (or 0.10) of this amount to the US government.

$$\$22,880 \times 0.10 = \$2288$$

After subtracting the tax, this leaves you with your **net income**, or **take-home pay**:

$$\$22,880 - \$2288 = \$20,592$$

Ok, so now let's figure out the yearly expenses. If monthly expenses are \$1500, then each year you will pay:

$$12 \times \$1500 = \$18,000$$

Finally, after the bills are paid, you can do what we want to with the remainder. This is our **discretionary budget**, and it is as good a measure as any as to how successful you are.

$$\$20,592 - \$18,000 = \$2592$$

Now keep that number \$2592 in mind. Let's see what kind of discretionary budget you will have if you take the new job. Since salary is, by definition, a yearly income amount, our first task is to compute and deduct the taxes. This time, you will pay \$2500 plus 15% of the difference between the salary and \$25,000.

$$\$2500 + (\$7000 \times 0.15) = \$3550$$

Therefore, the net income would be:

$$\$32,000 - \$3550 = \$28,450$$

It's a higher net income than you are currently making, but how will it stack up against the higher cost-of-living expenses in the city? Take the estimated monthly expenses of \$2000 and multiply it by 12 months:

$$12 \times \$2000 = \$24,000$$

Thus your new discretionary budget would be:

$$\$28,450 - \$24,000 = \$4450$$

That's definitely an improvement over \$2592. Maybe it's time to move to the big city and start advancing your career. However, you can see that it's not really that much more, so you probably shouldn't go out and buy a brand new car. Just wait until you get your first promotion to full manager.

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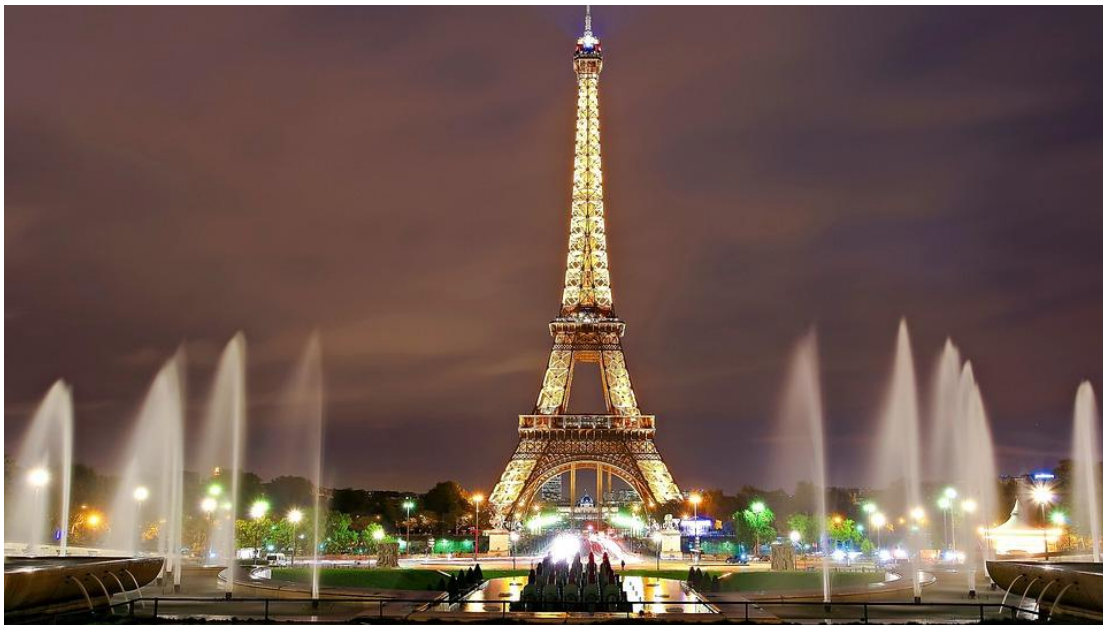
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MEASUREMENT

WHY IT MATTERS: MEASUREMENT



You've saved up for a trip to France with your French club. You have arrived safely and are ready to begin your adventures. There is just one problem—everything is described in terms of the metric system. The temperature you check to decide what to wear, the height of the Eiffel Tower you want to visit, the distance you are planning to travel for sightseeing, and even the amount of gasoline you need to purchase for your rental car are all presented in metric units.

- What should you wear if the temperature is going to be 18°C ?
- If the Eiffel Tower is 300 meters tall, is it taller than the Empire State Building at 1,250 feet?

- How long will it take you to drive from Paris to Bordeaux if it is 565 km from Paris?
- How many liters of gasoline will you need to buy to fill a gas tank that holds 13 gallons?

Have no fear. It may all seem very confusing, but it won't be for long. As you read about units of measurement in both U.S. customary units and metric units, it will all begin to make sense. After completing the module, we will revisit these questions to help you better plan your European excursion.

Learning Objectives

Units of Measurement

- Define units of length, weight, and capacity and convert from one to another.
- Perform arithmetic calculations on units of length, weight, and capacity.
- Solve application problems involving units of length, weight, and capacity.

Systems and scales of measurement

- Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and volume.
- Define the metric prefixes and use them to perform basic conversions among metric units.
- Solve application problems involving metric units of length, mass, and volume.
- State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
- Convert from one temperature scale to the other, using conversion formulas.

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INTRODUCTION: US UNITS OF MEASUREMENT

LEARNING OBJECTIVES

- Define units of length and convert from one to another.
- Perform arithmetic calculations on units of length.
- Solve application problems involving units of length.
- Define units of weight and convert from one to another.
- Perform arithmetic calculations on units of weight.
- Solve application problems involving units of weight.
- Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and volume.
- Define the metric prefixes and use them to perform basic conversions among metric units.
- Solve application problems involving metric units of length, mass, and volume.
- State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
- Convert from one temperature scale to the other, using conversion formulas.

Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the **metric system** and the **U.S. customary measurement system**.

In this section we will explore units for length, weight, and capacity, as well as solve problems that involve converting between different units of length, weight or capacity.

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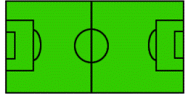
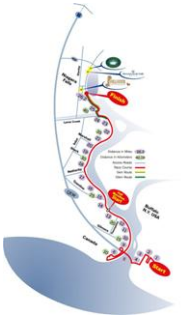
UNITS OF LENGTH

Suppose you want to purchase tubing for a project, and you see two signs in a hardware store: *\$1.88 for 2 feet* of tubing and *\$5.49 for 3 yards* of tubing. If both types of tubing will work equally well for your project, which is the better price? You need to know about two **units of measurement**, yards and feet, in order to determine the answer.

Length is the distance from one end of an object to the other end, or from one object to another. For example, the length of a letter-sized piece of paper is 11 inches. The system for measuring length in the United States is based on the four customary units of length: **inch**, **foot**, **yard**, and **mile**. Below are examples to show measurement in each of these units.



Unit	Description	Image
Inch/Inches	Some people donate their hair to be made into wigs for cancer patients who have lost hair as a result of treatment. One company requires hair donations to be at least 8 inches long.	
	Frame size of a bike: the distance from the center of the crank to the top of the seat tube. Frame size is usually measured in inches. This frame is 16 inches.	
Foot/Feet	Rugs are typically sold in standard lengths. One typical size is a rug that is 8 feet wide and 11 feet long. This is often described as an 8 by 11 rug.	

		
Yard/Yards	Soccer fields vary some in their size. An official field can be any length between 100 and 130 yards.	
Mile/Miles	A marathon is 26.2 miles long. One marathon route is shown in the map to the right.	

You can use any of these four U.S. customary measurement units to describe the length of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the length of a rug in feet rather than miles, and to describe a marathon in miles rather than inches.

You may need to convert between units of measurement. For example, you might want to express your height using feet and inches (5 feet 4 inches) or using only inches (64 inches). You need to know the unit equivalents in order to make these conversions between units.

The table below shows equivalents and conversion factors for the four customary units of measurement of length.

Unit Equivalents	Conversion Factors (longer to shorter units of measurement)	Conversion Factors (shorter to longer units of measurement)
1 foot = 12 inches	$\frac{12 \text{ inches}}{1 \text{ foot}}$	$\frac{1 \text{ foot}}{12 \text{ inches}}$
1 yard = 3 feet	$\frac{3 \text{ feet}}{1 \text{ yard}}$	$\frac{1 \text{ yard}}{3 \text{ feet}}$
1 mile = 5,280 feet	$\frac{5,280 \text{ feet}}{1 \text{ mile}}$	$\frac{1 \text{ mile}}{5,280 \text{ feet}}$

Note that each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all since it is the same as multiplying by 1; it just changes the units that you are using to measure.

Convert Between Different Units of Length

You can use the conversion factors to convert a measurement, such as feet, to another type of measurement, such as inches.

Note that there are many more inches for a measurement than there are feet for the same measurement, as feet is a longer unit of measurement. You could use the conversion factor $\frac{12 \text{ inches}}{1 \text{ foot}}$.

If a length is measured in feet, and youd like to convert the length to yards, you can think, I am converting from a shorter unit to a longer one, so the length in yards will be less than the length in feet. You could use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.

If a distance is measured in miles, and you want to know how many feet it is, you can think, I am converting from a longer unit of measurement to a shorter one, so the number of feet would be greater than the number of miles. You could use the conversion factor $\frac{5,280 \text{ feet}}{1 \text{ mile}}$.

You can use the **factor label method** (also known as **dimensional analysis**) to convert a length from one unit of measure to another using the conversion factors. In the factor label method, you multiply by unit fractions to convert a measurement from one unit to another. Study the example below to see how the factor label method can be used to convert $3\frac{1}{2}$ feet into an equivalent number of inches.

Example

How many inches are in $3\frac{1}{2}$ feet?

Answer

Begin by reasoning about your answer. Since a foot is longer than an inch, this means the answer would be greater than $3\frac{1}{2}$.

Find the conversion factor that compares inches and feet, with inches in the numerator, and multiply.

$$3\frac{1}{2} \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = ? \text{ inches}$$

Rewrite the mixed number as an improper fraction before multiplying.

$$\frac{7}{2} \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = ? \text{ inches}$$

You can cancel similar units when they appear in the numerator *and* the denominator. So here, cancel the similar units feet and foot. This eliminates this unit from the problem.

$$\frac{7}{2} \cancel{\text{feet}} \cdot \frac{12 \text{ inches}}{1 \cancel{\text{foot}}} = ? \text{ inches}$$

Rewrite as multiplication of numerators and denominators.

$$\frac{7 \cdot 12 \text{ inches}}{2} = \frac{84 \text{ inches}}{2} = 42 \text{ inches}$$

There are 42 inches in $3\frac{1}{2}$ feet.

Notice that by using the factor label method you can cancel the units out of the problem, just as if they were numbers. You can only cancel if the unit being cancelled is in both the numerator and denominator of the fractions you are multiplying.

In the problem above, you cancelled *feet* and *foot* leaving you with *inches*, which is what you were trying to find.

$$\frac{7}{2} \cancel{\text{feet}} \cdot \frac{12 \text{ inches}}{\cancel{1 \text{ foot}}} = ? \text{ inches}$$

What if you had used the wrong conversion factor?

$$\frac{7}{2} \text{ feet} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = ? \text{ inches?}$$

You could not cancel the feet because the unit is not the same in *both* the numerator and the denominator. So if you complete the computation, you would still have both feet and inches in the answer and no conversion would take place.

Here is another example of a length conversion using the factor label method.

Example

How many yards is 7 feet?

Answer

Start by reasoning about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. So your answer will be less than 7.

Find the conversion factor that compares feet and yards, with yards in the numerator.

$$7 \text{ feet} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = ? \text{ yards}$$

Cancel the similar units feet and feet leaving only yards.

$$7 \cancel{\text{ feet}} \cdot \frac{1 \text{ yard}}{3 \cancel{\text{ feet}}} = ? \text{ yards}$$

$$7 \cdot \frac{1 \text{ yard}}{3} = ? \text{ yards}$$

$$7 \text{ feet equals } 2\frac{1}{3} \text{ yards.}$$

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Apply Unit Conversions With Length

There are times when you will need to perform computations on measurements that are given in different units. For example, consider the tubing problem given earlier. You must decide which of the two options is a better price, and you have to compare prices given in different unit measurements.

In order to compare, you need to convert the measurements into one single, common unit of measurement. To be sure you have made the computation accurately, think about whether the unit you are converting to is

smaller or larger than the number you have. Its relative size will tell you whether the number you are trying to find is greater or lesser than the given number.

Example

An interior decorator needs border trim for a home she is wallpapering. She needs 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room. How many yards of border trim does she need?

Answer

You need to find the total length of border trim that is needed for all three rooms in the house. Since the measurements for each room are given in feet, you can add the numbers.

$$15 \text{ feet} + 30 \text{ feet} + 26 \text{ feet} = 71 \text{ feet}$$

How many yards is 71 feet?

Reason about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. Expect your answer to be less than 71. Use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$

$$\frac{71 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = ? \text{ yards}$$

$$\frac{71 \cancel{\text{ feet}}}{1} \cdot \frac{1 \text{ yard}}{3 \cancel{\text{ feet}}} = 23\frac{2}{3} \text{ yards}$$

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The next example uses the factor label method to solve a problem that requires converting from miles to feet.

Example

Two runners were comparing how much they had trained earlier that day. Jo said, According to my pedometer, I ran 8.3 miles. Alex said, That's a little more than what I ran. I ran 8.1 miles. How many more feet did Jo run than Alex?

Solution

You need to find the difference between the distance Jo ran and the distance Alex ran. Since both distances are given in the same unit, you can subtract and keep the unit the same.

$$8.3 \text{ miles} - 8.1 \text{ miles} = 0.2 \text{ mile}$$

$$0.2 \text{ mile} = \frac{2}{10} \text{ mile}$$

Since the problem asks for the difference in *feet*, you must convert from miles to feet. How many feet is 0.2 mile? Reason about the size of your answer. Since a mile is longer than a foot, the distance when expressed as feet will be a number greater than 0.2.

$$\frac{2}{10} \text{ mile} = \text{___ feet}$$

Use the conversion factor $\frac{5,280 \text{ feet}}{1 \text{ mile}}$.

$$\frac{2 \text{ mile}}{10} \cdot \frac{5,280 \text{ feet}}{1 \text{ mile}} = \text{___ feet}$$

$$\frac{2 \cancel{\text{ mile}}}{10} \cdot \frac{5,280 \text{ feet}}{1 \cancel{\text{ mile}}} = \text{___ feet}$$

$$\frac{2}{10} \cdot \frac{5,280 \text{ feet}}{1} = \underline{\hspace{1cm}} \text{ feet}$$

Multiply. Divide.

$$\frac{2 \bullet 5,280 \text{ feet}}{10 \bullet 1} = \underline{\hspace{1cm}} \text{ feet}$$

$$\frac{10,560 \text{ feet}}{10} = \underline{\hspace{1cm}} \text{ feet}$$

$$\frac{10,560 \text{ feet}}{10} = 1,056 \text{ feet}$$

Jo ran 1,056 feet further than Alex.

In the next example we show how to compare the price of two different kinds of tubing for a project you are making. One type of tubing is given in cost per yards, and the other is given in cost per foot. It is easier to make a comparison when the units are the same, so we convert one price into the same units as the other. For problems like this, it doesn't matter which cost you convert, either one will work.

Example

You are walking through a hardware store and notice two sales on tubing.

3 yards of Tubing A costs \$5.49.

Tubing B sells for \$1.88 for 2 feet.

Either tubing is acceptable for your project. Which tubing is less expensive?

Answer

Find the unit price for each tubing. This will make it easier to compare.

Tubing A

Find the cost per yard of Tubing A by dividing the cost of 3 yards of the tubing by 3.

3 yards = \$5.49

$$\frac{5.49 \div 3}{3 \text{ yards} \div 3} = \frac{\$1.83}{1 \text{ yard}}$$

Tubing B is sold by the foot. Find the cost per foot by dividing \$1.88 by 2 feet.

Tubing B

2 feet = \$1.88

$$\frac{1.88 \div 2}{2 \text{ feet} \div 2} = \frac{\$0.94}{1 \text{ foot}}$$

To compare the prices, you need to have the same unit of measure.

Use the conversion factor $\frac{3 \text{ feet}}{1 \text{ yard}}$, cancel and multiply.

$$\frac{\$0.94}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yard}} = \frac{\$ \hspace{1cm}}{\text{yard}}$$

$$\frac{\$0.94}{\cancel{1 \text{ foot}}} \cdot \frac{\cancel{3 \text{ feet}}}{1 \text{ yard}} = \frac{\$2.82}{1 \text{ yard}}$$

\$2.82 per yard

Compare prices for 1 yard of each tubing.

Tubing A: \$1.83 per yard

Tubing B: \$2.82 per yard

Tubing A is less expensive than Tubing B.

In the problem above, you could also have found the price per foot for each kind of tubing and compared the unit prices of each per foot.

You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic. The four basic units of measurement that are used in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths.

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UNITS OF WEIGHT

When you mention how heavy or light an object is, you are referring to its weight. In the U.S. customary system of measurement, weight is measured in ounces, pounds, and tons. These measurements actually refer to how much the gravitational force of the Earth pulls on the object. Like other units of measurement, you can convert between these units and you sometimes need to do this to solve problems.

The grocery store sells a 36 ounce canister of ground coffee for \$14, and sells bulk coffee for \$9 per pound. Which is the better deal? To answer this question, you need to understand the relationship between ounces and pounds.

You often use the word **weight** to describe how heavy or light an object or person is. Weight is measured in the U.S. customary system using three units: ounces, pounds, and tons. An **ounce** is the smallest unit for measuring weight, a **pound** is a larger unit, and a **ton** is the largest unit.

Whales are some of the largest animals in the world. Some species can reach weights of up to 200 tons—that's equal to 400,000 pounds.



Meat is a product that is typically sold by the pound. One pound of ground beef makes about four hamburger patties.



Ounces are used to measure lighter objects. A stack of 11 pennies is equal to about one ounce.



You can use any of the customary measurement units to describe the weight of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the weight of a human being in pounds rather than tons. It makes more sense to describe the weight of a car in tons rather than ounces.

1 pound = 16 ounces

$$\frac{16 \text{ ounces}}{1 \text{ pound}}$$

Converting Between Units of Weight

Four ounces is a typical serving size of meat. Since meat is sold by the pound, you might want to convert the weight of a package of meat from pounds to ounces in order to determine how many servings are contained in a package of meat.

The weight capacity of a truck is often provided in tons. You might need to convert pounds into tons if you are trying to determine whether a truck can safely transport a big shipment of heavy materials.

The table below shows the unit conversions and conversion factors that are used to make conversions between customary units of weight.

Unit Equivalents	Conversion Factors (heavier to lighter units of measurement)	Conversion Factors (lighter to heavier units of measurement)
1 pound = 16 ounces	$\frac{16 \text{ ounces}}{1 \text{ pound}}$	$\frac{1 \text{ pound}}{16 \text{ ounces}}$
1 ton = 2000 pounds	$\frac{2000 \text{ pounds}}{1 \text{ ton}}$	$\frac{1 \text{ ton}}{2000 \text{ pounds}}$

You can use the *factor label method* to convert one customary unit of weight to another customary unit of weight. This method uses conversion factors, which allow you to cancel units to end up with your desired unit of measurement.

Each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all, since it is the same as multiplying by 1. It just changes the units that you are using to measure it in.

Two examples illustrating the factor label method are shown below.

Exercises

How many ounces are in $2\frac{1}{4}$ pounds?

Answer

Begin by reasoning about your answer. Since a pound is heavier than an ounce, expect your answer to be a number greater than $2\frac{1}{4}$.

$$2\frac{1}{4} \text{ pounds} = \underline{\hspace{1cm}} \text{ ounces}$$

Multiply by the conversion factor that relates ounces and pounds: $\frac{16 \text{ ounces}}{1 \text{ pound}}$.

$$2\frac{1}{4} \text{ pounds} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} = \underline{\hspace{1cm}} \text{ ounces}$$

Write the mixed number as an improper fraction.

The common unit pound can be cancelled because it appears in both the numerator and denominator.

$$\frac{9 \text{ pounds}}{4} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} = \underline{\hspace{1cm}} \text{ ounces}$$

$$\frac{9 \cancel{\text{ pounds}}}{4} \cdot \frac{16 \text{ ounces}}{1 \cancel{\text{ pound}}} = \underline{\hspace{1cm}} \text{ ounces}$$

Multiply and simplify.

$$\frac{9}{4} \cdot \frac{16 \text{ ounces}}{1} = \underline{\hspace{1cm}} \text{ ounces}$$

$$\frac{9 \cdot 16 \text{ ounces}}{4 \cdot 1} = \underline{\hspace{1cm}} \text{ ounces}$$

$$\frac{144 \text{ ounces}}{4} = \underline{\hspace{1cm}} \text{ ounces}$$

$$\frac{144 \text{ ounces}}{4} = 36 \text{ ounces}$$

There are 36 ounces in $2\frac{1}{4}$ pounds.

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Example

How many tons is 6,500 pounds?

Answer

Begin by reasoning about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 6,500.

$$6,500 \text{ pounds} = \underline{\hspace{1cm}} \text{ tons}$$

Multiply by the conversion factor that relates tons to pounds: $\frac{1 \text{ ton}}{2,000 \text{ pounds}}$.

Apply the Factor Label method.

Multiply and simplify.

$$6,500 \text{ pounds} \cdot \frac{1 \text{ ton}}{2,000 \text{ pounds}} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{6,500 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2,000 \text{ pounds}} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{6,500 \cancel{\text{ pounds}}}{1} \cdot \frac{1 \text{ ton}}{2,000 \cancel{\text{ pounds}}} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{6,500}{1} \cdot \frac{1 \text{ ton}}{2,000} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{6,500 \text{ pounds}}{2,000} = 3\frac{1}{4} \text{ tons}$$

6,500 pounds is equal to $3\frac{1}{4}$ tons.

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Applications of Unit Conversions With Weight

There are times when you need to perform calculations on measurements that are given in different units. To solve these problems, you need to convert one of the measurements to the same unit of measurement as the other measurement.

Think about whether the unit you are converting to is smaller or larger than the unit you are converting from. This will help you be sure that you are making the right computation. You can use the factor label method to make the conversion from one unit to another.

Here is an example of a problem that requires converting between units.

Example

A municipal trash facility allows a person to throw away a maximum of 30 pounds of trash per week. Last week, 140 people threw away the maximum allowable trash. How many tons of trash did this equal?
Solution

Determine the total trash for the week expressed in pounds.

If 140 people each throw away 30 pounds, you can find the total by multiplying.

$$140 \cdot 30 \text{ pounds} = 4,200 \text{ pounds}$$

Then convert 4,200 pounds to tons. Reason about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 4,200.

$$4,200 \text{ pounds} = \underline{\hspace{1cm}} \text{ tons}$$

Find the conversion factor appropriate for the situation:

$$\begin{array}{r} 1 \text{ ton} \\ \hline 2,000 \text{ pounds} \end{array}$$
$$\frac{4,200 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2,000 \text{ pounds}} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{4,200 \cancel{\text{ pounds}}}{1} \cdot \frac{1 \text{ ton}}{2,000 \cancel{\text{ pounds}}} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{4,200}{1} \cdot \frac{1 \text{ ton}}{2,000} = \underline{\hspace{1cm}} \text{ tons}$$

Multiply and simplify.

$$\frac{4,200 \cdot 1 \text{ ton}}{1 \cdot 2,000} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{4,200 \text{ ton}}{2,000} = \underline{\hspace{1cm}} \text{ tons}$$

$$\frac{4,200 \text{ ton}}{2,000} = 2\frac{1}{10} \text{ tons}$$

The total amount of trash generated is $2\frac{1}{10}$ tons.

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Let's revisit the coffee price problem that was posed earlier. We can use unit conversion to solve this problem.

Example

The grocery store sells a 36 ounce canister of ground coffee for \$14, and sells bulk coffee for \$7 per pound. Which is the better deal?

Solution

Since canister pricing is for ounces, convert the weight of the canister to pounds.

First use the factor label method to convert ounces to pounds.

36 ounces = ____ pounds

$$\frac{36 \text{ ounces}}{1} \cdot \frac{1 \text{ pound}}{16 \text{ ounces}} = \text{____ pound}$$

$$\frac{36 \cancel{\text{ ounces}}}{1} \cdot \frac{1 \text{ pound}}{16 \cancel{\text{ ounces}}} = \text{____ pound}$$

$$\frac{36}{1} \cdot \frac{1 \text{ pound}}{16} = 2\frac{1}{4} \text{ pounds}$$

Now calculate the price per pound by dividing.

$$\frac{14}{2\frac{1}{4} \text{ pounds}}$$

$$\frac{14}{2\frac{1}{4} \text{ pounds}} \approx \$\$6.22 \text{ per pound}$$

The canister is a better deal at \$6.22 per pound.

Try It Now

The average weight of a northern bluefin tuna is 1,800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern bluefin tuna?

Answer

3200 lbs.

Summary

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2,000 pounds. While an objects weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people.

Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

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UNITS OF CAPACITY

Capacity is the amount of liquid (or other pourable substance) that an object can hold when its full. When a liquid, such as milk, is being described in gallons or quarts, this is a measure of capacity.

Understanding units of capacity can help you solve problems like this: Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How many cups of soup did they have all together?

There are five main units for measuring capacity in the U.S. customary measurement system. The smallest unit of measurement is a **fluid ounce**. Ounce is also used as a measure of weight, so it is important to use the word fluid with ounce when you are talking about capacity. Sometimes the prefix fluid is not used when it is clear from the context that the measurement is capacity, not weight.

The other units of capacity in the customary system are the **cup**, **pint**, **quart**, and **gallon**. The table below describes each unit of capacity and provides an example to illustrate the size of the unit of measurement.

<p>Fluid Ounce</p> <p>A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62°F weighs about one ounce. The amount of liquid medicine is often measured in fluid ounces.</p>	
<p>Cup</p> <p>A unit equal to 8 fluid ounces. The capacity of a standard measuring cup is one cup.</p>	
<p>Pint</p> <p>A unit equal to 16 fluid ounces, or 2 cups. The capacity of a carton of ice cream is often measured in pints.</p>	
<p>Quart</p>	

A unit equal to 32 fluid ounces, or 4 cups. You often see quarts of milk being sold in the supermarket.



Gallon

A unit equal to 4 quarts, or 128 fluid ounces. When you fill up your car with gasoline, the price of gas is often listed in dollars per gallon.



You can use any of these five measurement units to describe the capacity of an object, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the capacity of a swimming pool in gallons and the capacity of an expensive perfume in fluid ounces.

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Sometimes you will need to convert between units of measurement. For example, you might want to express 5 gallons of lemonade in cups if you are trying to determine how many 8-fluid ounce servings the amount of lemonade would yield.

The table below shows some of the most common equivalents and conversion factors for the five customary units of measurement of capacity.

Unit Equivalents	Conversion Factors (heavier to lighter units of measurement)	Conversion Factors (lighter to heavier units of measurement)
1 cup = 8 fluid ounces	$\frac{1 \text{ cup}}{8 \text{ fluid ounces}}$	$\frac{8 \text{ fluid ounces}}{1 \text{ cup}}$
1 pint = 2 cups	$\frac{1 \text{ pint}}{2 \text{ cups}}$	$\frac{2 \text{ cups}}{1 \text{ pint}}$
1 quart = 2 pints	$\frac{1 \text{ quart}}{2 \text{ pints}}$	$\frac{2 \text{ pints}}{1 \text{ quart}}$
1 quart = 4 cups	$\frac{1 \text{ quart}}{4 \text{ cups}}$	$\frac{4 \text{ cups}}{1 \text{ quart}}$
1 gallon = 4		

quarts	$\frac{1 \text{ gallon}}{4 \text{ quarts}}$	$\frac{4 \text{ quarts}}{1 \text{ gallon}}$
1 gallon = 16 cups	$\frac{1 \text{ gallon}}{16 \text{ cups}}$	$\frac{16 \text{ cups}}{1 \text{ gallon}}$

Converting Between Units of Capacity

As with converting units of length and weight, you can use the factor label method to convert from one unit of capacity to another. An example of this method is shown below.

Example

How many pints is $2\frac{3}{4}$ gallons?

Answer

Begin by reasoning about your answer. Since a gallon is larger than a pint, expect the answer in pints to be a number greater than $2\frac{3}{4}$.

$$2\frac{3}{4} \text{ gallons} = \text{___ pints}$$

The table above does not contain a conversion factor for gallons and pints, so you cannot convert it in one step. However, you can use quarts as an intermediate unit, as shown here.

Set up the equation so that two sets of labels cancel gallons and quarts.

$$\frac{11 \text{ gallons}}{4} \cdot \frac{4 \text{ quarts}}{1 \text{ gallon}} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} = \text{___ pints}$$

$$\frac{11 \cancel{\text{ gallons}}}{4} \cdot \frac{4 \cancel{\text{ quarts}}}{1 \cancel{\text{ gallon}}} \cdot \frac{2 \text{ pints}}{1 \cancel{\text{ quart}}} = \text{___ pints}$$

$$\frac{11}{4} \cdot 4 \cdot \frac{2 \text{ pints}}{1} = \text{___ pints}$$

Multiply and simplify.

$$\frac{11 \cdot 4 \cdot 2 \text{ pints}}{4} = \text{___ pints}$$

$$\frac{88 \text{ pints}}{4} = 22 \text{ pints}$$

$$2\frac{3}{4} \text{ gallons is } 22 \text{ pints.}$$

Example

How many gallons is 32 fluid ounces?

Answer

Begin by reasoning about your answer. Since gallons is a larger unit than fluid ounces, expect the answer to be less than 32.

$$32 \text{ fluid ounces} = \text{___ gallons}$$

The table above does not contain a conversion factor for gallons and fluid ounces, so you cannot convert it in one step. Use a series of intermediate units, as shown here.

$$\frac{32 \text{ fl oz}}{1} \cdot \frac{1 \text{ cup}}{8 \text{ fl oz}} \cdot \frac{1 \text{ qt}}{2 \text{ pt}} \cdot \frac{1 \text{ gal}}{4 \text{ qt}} = \text{ ______ gal}$$

Cancel units that appear in both the numerator and denominator.

$$\frac{32 \cancel{\text{ fl oz}}}{1} \cdot \frac{1 \cancel{\text{ cup}}}{8 \cancel{\text{ fl oz}}} \cdot \frac{1 \cancel{\text{ qt}}}{2 \cancel{\text{ pt}}} \cdot \frac{1 \text{ gal}}{4 \cancel{\text{ qt}}} = \text{ ______ gal}$$

$$\frac{32}{1} \cdot \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1 \text{ gal}}{4} = \text{ ______ gal}$$

Multiply and simplify.

$$\frac{32 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \text{ gal}}{1 \cdot 8 \cdot 2 \cdot 2 \cdot 4} = \text{ ______ gal}$$

$$\frac{32 \text{ gal}}{128} = \frac{1}{4} \text{ gal}$$

32 fluid ounces is the same as $\frac{1}{4}$ gallon.

Try It Now

Find the sum of 4 gallons and 2 pints. Express your answer in cups.

Answer

4 gallons + 2 pints = 64 cups + 4 cups = 68 cups

Visit this page in your course online to practice before taking the quiz.

Visit this page in your course online to practice before taking the quiz.

Applying Unit Conversions

There are times when you will need to combine measurements that are given in different units. In order to do this, you need to convert first so that the units are the same.

Consider the situation posed earlier in this topic.

Exercises

Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How much soup did they have total?

Answer

Since the problem asks for the total amount of soup, you must add the three quantities. Before adding, you must convert the quantities to the same unit.

The problem does not require a particular unit, so you can choose. Cups might be the easiest computation.

1 quart + 3 pints + 9 cups

This is given in the table of equivalents.

1 quart = 4 cups

Use the factor label method to convert pints to cups.

$$\frac{3 \text{ pints}}{1} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = \text{--- cups}$$

$$\frac{\cancel{3 \text{ pints}}}{1} \cdot \frac{2 \text{ cups}}{\cancel{1 \text{ pint}}} = 6 \text{ cups}$$

Add the 3 quantities.

$$4 \text{ cups} + 6 \text{ cups} + 9 \text{ cups} = 19 \text{ cups}$$

There are 19 cups of soup for the dinner.

Exercises

Natasha is making lemonade to bring to the beach. She has two containers. One holds one gallon and the other holds 2 quarts. If she fills both containers, how many cups of lemonade will she have?

Answer

This problem requires you to find the sum of the capacity of each container and then convert that sum to cups.

$$1 \text{ gallon} + 2 \text{ quarts} = \text{--- cups}$$

First, find the sum in quarts. 1 gallon is equal to 4 quarts.

$$4 \text{ quarts} + 2 \text{ quarts} = 6 \text{ quarts}$$

Since the problem asks for the capacity in cups, convert 6 quarts to cups.

Cancel units that appear in both the numerator and denominator.

Multiply.

$$\frac{6 \text{ quarts}}{1} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = \text{--- cups}$$

$$\frac{\cancel{6 \text{ quarts}}}{1} \cdot \frac{\cancel{2 \text{ pints}}}{\cancel{1 \text{ quart}}} \cdot \frac{2 \text{ cups}}{\cancel{1 \text{ pint}}} = \text{--- cups}$$

$$6 \times 2 \times 2 = 24 \text{ cups}$$

Natasha will have 24 cups of lemonade.

Another way to work the problem above would be to first change 1 gallon to 16 cups and change 2 quarts to 8 cups. Then add: $16 + 8 = 24$ cups.

In the following video we provide another example of using unit conversions to solve a problem. We show how to find the number of lemons needed to make a pie, given that each lemon yields about 4 tablespoons of juice.

Watch this video online: <https://youtu.be/4NJ6oqXflbE>

Try It Now

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2-gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

Answer

The chili will only fit in the 2 gallon pot

In the following ~10 minute video, we provide a mini-lesson that covers US measurements for length, weight, and capacity, and how to convert between larger and smaller units for each type. This is a good summary of the concepts covered in the US Units of Measurement section of this module.

Watch this video online: <https://youtu.be/ozSnWr4do5o>

Summary

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

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INTRODUCTION: SYSTEMS AND SCALES OF MEASUREMENT

Learning Objectives

In this lesson you will learn how to do the following:

- Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and volume
- Define the metric prefixes and use them to perform basic conversions among metric units
- Solve application problems using metric units
- State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
- Convert from one temperature scale to the other, using conversion formulas

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like kilometer, liter, and milligram to measure the length, volume, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of 10 is not present in the U.S. customary system there are 12 inches in a foot, and 3 feet in a yard and 5,280 feet in a mile!

So, what if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

In this section we will discover the basic units used in the metric system, and show how to convert between them. We will also explore temperature scales. In the United States, temperatures are usually measured using the **Fahrenheit** scale, while most countries that use the metric system use the **Celsius** scale to record temperatures. Learning about the different scales, including how to convert between them will help you figure out what the weather is going to be like, no matter which country you find yourself in.

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METRIC SYSTEM BASICS

What Is Metric?

The metric system uses units such as **meter**, **liter**, and **gram** to measure length, liquid volume, and mass, just as the U.S. customary system uses feet, quarts, and ounces to measure these.

In addition to the difference in the basic units, the metric system is based on 10s, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word meter is part of all of these units.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1,000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of getting bigger or smaller by 10 is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard, and 16 ounces equals 1 pound.

Length, Mass, and Volume

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.




Length	Mass	Volume
<i>basic units</i>		
meter	gram	liter
<i>other units you may see</i>		
kilometer	kilogram	dekaliter
centimeter	centigram	centiliter
millimeter	milligram	milliliter

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram.

Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word mass is used here instead of weight. In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an objects weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the objects mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeablybut it is worth noting that there is a difference!

Finally, the basic metric unit of volume is the liter. A liter is slightly larger than a quart.

		
The handle of a shovel is about 1 meter.	A paperclip weighs about 1 gram.	A medium-sized container of milk is about 1 liter.

Though it is rarely necessary to convert between the customary and metric systems, sometimes it helps to have a mental image of how large or small some units are. The table below shows the relationship between some common units in both systems.

	Common Measurements in Customary and Metric Systems
<i>Length</i>	1 centimeter is a little less than half an inch.
	1.6 kilometers is about 1 mile.
	1 meter is about 3 inches longer than 1 yard.
<i>Mass</i>	1 kilogram is a little more than 2 pounds.
	28 grams is about the same as 1 ounce.
<i>Volume</i>	1 liter is a little more than 1 quart.
	4 liters is a little more than 1 gallon.

Prefixes in the Metric System

The metric system is a base 10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the **prefix**. To tell whether the unit is measuring length, mass, or volume, you look at the base.

Prefixes in the Metric System						
<i>kilo-</i>	<i>hecto-</i>	<i>deka-</i>		<i>deci-</i>	<i>centi-</i>	<i>milli-</i>

			meter			
			gram			
			liter			
1,000 times larger than base unit	100 times larger than base unit	10 times larger than base unit	base units	10 times smaller than base unit	100 times smaller than base unit	1,000 times smaller than base unit

Using this table as a reference, you can see the following:

- A kilogram is 1,000 times larger than one gram (so 1 kilogram = 1,000 grams).
- A centimeter is 100 times smaller than one meter (so 1 meter = 100 centimeters).
- A dekaliter is 10 times larger than one liter (so 1 dekaliter = 10 liters).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

Measuring Mass in the Metric System						
kilogram	hectogram	dekagram	gram	decigram	centigram	milligram
(kg)	(hg)	(dag)	(g)	(dg)	(cg)	(mg)
1,000 grams	100 grams	10 grams	gram	0.1 gram	0.01 gram	0.001 gram

Since the prefixes remain constant through the metric system, you could create similar charts for length and volume. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or volume (liter).

Try It Now

Which of the following sets of three units are all metric measurements of **length**?

- A) inch, foot, yard
- B) kilometer, centimeter, millimeter
- C) kilogram, gram, centigram
- D) kilometer, foot, decimeter

Answer

B) kilometer, centimeter, millimeter

All of these measurements are from the metric system. You can tell they are measurements of length because they all contain the word meter.

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Converting Units Up and Down the Metric Scale

Converting between metric units of measure requires knowledge of the metric prefixes and an understanding of the decimal system that's about it.

For instance, you can figure out how many centigrams are in one dekagram by using the table above. One dekagram is larger than one centigram, so you expect that one dekagram will equal many centigrams.

In the table, each unit is 10 times larger than the one to its immediate right. This means that 1 dekagram = 10 grams; 10 grams = 100 decigrams; and 100 decigrams = 1,000 centigrams. So, 1 dekagram = 1,000 centigrams.

Example

How many milligrams are in one decigram?

Answer

Identify locations of milligrams and decigrams.

kg	hg	dag	g	dg	cg	mg
^	^					

Decigrams (dg) are larger than milligrams (mg), so you expect there to be many mg in one dg. Dg is 10 times larger than a cg, and a cg is 10 times larger than a mg.

×10	×10					
kg	hg	dag	g	dg	cg	mg
↓	↑					
→						

Since you are going from a larger unit to a smaller unit, multiply.

Multiply: $1 \cdot 10 \cdot 10$, to find the number of milligrams in one decigram.

$$1 \text{ dg} \cdot 10 \cdot 10 = 100 \text{ mg}$$

There are 100 milligrams (mg) in 1 decigram (dg).

Try It Now

Convert 3,085 milligrams to grams.

Answer

One gram is 1,000 times larger than a milligram, so you can move the decimal point in 3,085 three places to the left.

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Example

Convert 1 centimeter to kilometers.

Answer

Identify locations of kilometers and centimeters.

km	hm	dam	m	dm	cm	mm
^					^	

Kilometers (km) are larger than centimeters (cm), so you expect there to be less than one km in a cm. Cm is 10 times smaller than a dm; a dm is 10 times smaller than a m, etc. Since you are going from a smaller unit to a larger unit, divide.

÷10	÷10	÷10	÷10	÷10		
km	hm	dam	m	dm	cm	mm
^	←	←	←	←	^	

Divide: $1 \div 10 \div 10 \div 10 \div 10 \div 10$, to find the number of kilometers in one centimeter.

$1 \text{ cm} \div 10 \div 10 \div 10 \div 10 \div 10 = 0.00001 \text{ km}$

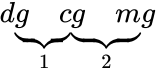
1 centimeter (cm) = 0.00001 kilometers (km).

Try It Now

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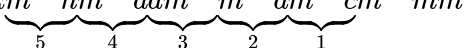
Once you begin to understand the metric system, you can use a shortcut to convert among different metric units. The size of metric units increases tenfold as you go up the metric scale. The decimal system works the same way: a tenth is 10 times larger than a hundredth; a hundredth is 10 times larger than a thousandth, etc. By applying what you know about decimals to the metric system, converting among units is as simple as moving decimal points.

Here is the first problem from above: How many milligrams are in one decigram? You can recreate the order of the metric units as shown below:

kg hg dag g dg cg mg


This question asks you to start with 1 decigram and convert that to milligrams. As shown above, milligrams is two places to the right of decigrams. You can just move the decimal point two places to the right to convert decigrams to milligrams: $1 \text{ dg} = 1 \overset{0}{\underset{1}{\text{}}} \overset{0}{\underset{2}{\text{}}} . \text{ mg}$.

The same method works when you are converting from a smaller to a larger unit, as in the problem: Convert 1 centimeter to kilometers.

km hm dam m dm cm mm


Note that instead of moving to the right, you are now moving to the left so the decimal point must do the same:

$$1 \text{ cm} = 0.\underbrace{0\ 0\ 0\ 0\ 1}_{5\ 4\ 3\ 2\ 1} \text{ km.}$$

Try It Now

How many milliliters are in 1 liter?

Answer

There are 10 milliliters in a centiliter, 10 centiliters in a deciliter, and 10 deciliters in a liter. Multiply: $10 \cdot 10 \cdot 10$, to find the number of milliliters in a liter, 1,000.
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Factor Label Method

There is yet another method that you can use to convert metric measurements the **factor label method**. You used this method when you were converting measurement units within the U.S. customary system.

The factor label method works the same in the metric system; it relies on the use of unit fractions and the cancelling of intermediate units. The table below shows some of the **unit equivalents** and **unit fractions** for length in the metric system. (You should notice that all of the unit fractions contain a factor of 10. Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.)

Also, notice that two new prefixes have been added here: mega- (which is very big) and micro- (which is very small).

Unit Equivalents	Conversion Factors	
1 meter = 1,000,000 micrometers	$\frac{1 \text{ m}}{1,000,000 \text{ } \mu\text{m}}$	$\frac{1,000,000 \text{ } \mu\text{m}}{1 \text{ m}}$
1 meter = 1,000 millimeters	$\frac{1 \text{ m}}{1,000 \text{ mm}}$	$\frac{1,000 \text{ mm}}{1 \text{ m}}$
1 meter = 100 centimeters	$\frac{1 \text{ m}}{100 \text{ cm}}$	$\frac{100 \text{ cm}}{1 \text{ m}}$
1 meter = 10 decimeters	$\frac{1 \text{ m}}{10 \text{ dm}}$	$\frac{10 \text{ dm}}{1 \text{ m}}$
1 dekameter = 10 meters	$\frac{1 \text{ dam}}{10 \text{ m}}$	$\frac{10 \text{ m}}{1 \text{ dam}}$
1 hectometer = 100 meters	$\frac{1 \text{ hm}}{100 \text{ m}}$	$\frac{100 \text{ m}}{1 \text{ hm}}$
1 kilometer = 1,000 meters	$\frac{1 \text{ km}}{1,000 \text{ m}}$	$\frac{1,000 \text{ m}}{1 \text{ km}}$

1 megameter = 1,000,000 meters

$$\frac{1 \text{ Mm}}{1,000,000 \text{ m}}$$

$$\frac{1,000,000 \text{ m}}{1 \text{ Mm}}$$

When applying the factor label method in the metric system, be sure to check that you are not skipping over any intermediate units of measurement!

Example

Convert 7,225 centimeters to meters.

Answer

Meters is larger than centimeters, so you expect your answer to be less than 7,225.

$$7,225 \text{ cm} = \underline{\hspace{1cm}} \text{ m}$$

Using the factor label method, write 7,225 cm as a fraction and use unit fractions to convert it to m.

$$\frac{7,225 \text{ cm}}{1} \cdot \frac{1 \text{ m}}{100 \text{ cm}} = \underline{\hspace{1cm}} \text{ m}$$

Cancel similar units, multiply, and simplify.

$$\frac{7,225 \cancel{\text{ cm}}}{1} \cdot \frac{1 \text{ m}}{100 \cancel{\text{ cm}}} = \underline{\hspace{1cm}} \text{ m}$$

$$\frac{7,225}{1} \cdot \frac{1 \text{ m}}{100} = \frac{7,225}{100} \text{ m}$$

$$\frac{7,225 \text{ m}}{100} = 72.25 \text{ m}$$

$$7,225 \text{ centimeters} = 72.25 \text{ meters}$$

Try It Now

Convert 32.5 kilometers to meters.

Answer

32,500 meters

$$\frac{32.5 \text{ km}}{1} \cdot \frac{1,000 \text{ m}}{1 \text{ km}} = \frac{32,500 \text{ m}}{1}$$

The km units cancel, leaving the answer in m.

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Now that you have seen how to convert among metric measurements in multiple ways, let's revisit the problem posed earlier.

Example

If you have a prescription for 5,000 mg of medicine, and upon getting it filled, the dosage reads 5g of medicine, did the pharmacist make a mistake?

Answer

Convert mg to g.

$$\begin{aligned} 5,000 \text{ mg} &= \frac{1 \text{ g}}{1,000 \text{ mg}} \cdot 5,000 \text{ mg} \\ &= \frac{5,000 \cancel{\text{mg}}}{1} \cdot \frac{1 \text{ g}}{1,000 \cancel{\text{mg}}} \\ &= \frac{5,000 \cdot 1 \text{ g}}{1 \cdot 1,000} = \frac{5,000 \text{ g}}{1,000} \\ &= 5 \text{ g} \end{aligned}$$

5 g = 5,000 mg, so the pharmacist did not make a mistake.

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APPLICATIONS OF METRIC CONVERSIONS

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

TIP: To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

Understanding Context and Performing Conversions

The first step in solving any real-world problem is to understand its context. This will help you figure out what kinds of solutions are reasonable (and the problem itself may give you clues about what types of conversions are necessary). Here is an example.

Example

Marcus bought a 2 meter board, and cut off a piece 1 meter and 35 cm long. How much board is left?

Answer

To answer this question, we will need to subtract.

First convert all measurements to one unit. Here we will convert to centimeters.

2 meters – 1 meter and 35 cm

Use the factor label method and unit fractions to convert from meters to centimeters.

$$\frac{2 \text{ m}}{1} \cdot \frac{100 \text{ cm}}{1 \text{ m}} = \text{cm}$$

Cancel, multiply, and solve.

Convert the 1 meter to centimeters, and combine with the additional 35 centimeters.

Subtract the cut length from the original board length.

$$\frac{2 \cancel{\text{ m}}}{1} \cdot \frac{100 \text{ cm}}{1 \cancel{\text{ m}}} = \text{cm}$$

$$\frac{200 \text{ cm}}{1} = 200 \text{ cm}$$

1 meter + 35 cm

100 cm + 35 cm

135 cm

200 cm – 135 cm

65 cm

There is 65 cm of board left.

An example with a different context, but still requiring conversions, is shown below.

Example

A faucet drips 10 ml every minute. How much water will be wasted in a week?

Answer

Start by calculating how much water will be used in a week using the factor label method to convert the time units.

$$\frac{10 \text{ ml}}{1 \text{ minute}} \cdot \frac{60 \text{ minute}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{7 \text{ days}}{1 \text{ week}}$$

Cancel, multiply and solve.

$$\frac{10 \text{ ml}}{1 \cancel{\text{ minute}}} \cdot \frac{60 \cancel{\text{ minute}}}{1 \cancel{\text{ hour}}} \cdot \frac{24 \cancel{\text{ hours}}}{1 \cancel{\text{ day}}} \cdot \frac{7 \cancel{\text{ days}}}{1 \text{ week}}$$

$$10 \cdot 60 \cdot 24 \cdot 7 \text{ ml}$$

$$1 \cdot 1 \cdot 1 \cdot 1 \text{ week}$$

To give a more useable answer, convert this into liters.

$$\frac{100800 \text{ ml}}{1 \text{ week}}$$

Cancel, multiply and solve.

$$\frac{100800 \text{ ml}}{1 \text{ week}} \cdot \frac{1 \text{ L}}{1000 \text{ ml}}$$

$$\frac{100800 \cancel{\text{ ml}}}{1 \text{ week}} \cdot \frac{1 \text{ L}}{1000 \cancel{\text{ ml}}}$$

$$\frac{100800 \text{ L}}{1000 \text{ week}} = 100.8 \frac{\text{L}}{\text{week}}$$

$$\frac{100800 \text{ L}}{1000 \text{ week}} = 100.8 \frac{\text{L}}{\text{week}}$$

$$\frac{100800 \text{ L}}{1000 \text{ week}} = 100.8 \frac{\text{L}}{\text{week}}$$

The faucet wastes about 100.8 liters each week.

This problem asked for the difference between two quantities. The easiest way to find this is to convert one quantity so that both quantities are measured in the same unit, and then subtract one from the other.

Try It Now

A bread recipe calls for 600 g of flour. How many kilograms of flour would you need to make 5 loaves?

Answer

Multiplying 600 g per loaf by the 5 loaves,

$$600\text{g} \cdot 5 = 3000\text{g}$$

Using factor labels or the move the decimal method, convert this to 3 kilograms.

You will need 3 kg of flour to make 5 loaves.

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Checking your Conversions

Sometimes it is a good idea to check your conversions using a second method. This usually helps you catch any errors that you may make, such as using the wrong unit fractions or moving the decimal point the wrong way.

Example

A bottle contains 1.5 liters of a beverage. How many 250 mL servings can be made from that bottle?

Answer

To answer the question, you will need to divide 1.5 liters by 250 milliliters. To do this, convert both to the same unit. You could convert either measurement.

$$1.5 \text{ L} \div 250 \text{ mL}$$

Convert 250 mL to liters

$$250 \text{ mL} = \frac{\quad}{1000} \text{ L}$$

$$\frac{250 \text{ mL}}{1000 \text{ mL}} \cdot \frac{1 \text{ L}}{1000 \text{ mL}} = \frac{\quad}{1000} \text{ L}$$

$$1000 = 0.25 \text{ L}$$

Now we can divide using the converted measurement

$$1.5 \text{ L} \div 250 \text{ mL} = \frac{1.5 \text{ L}}{250 \text{ mL}} = \frac{1.5 \text{ L}}{0.25 \text{ L}}$$

$$\frac{1.5 \text{ L}}{0.25 \text{ L}} = 6$$

The bottle holds 6 servings.

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the move the decimal method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

Summary

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo-, hecto-, deka-, deci-,

centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one.

This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

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TEMPERATURE SCALES

Turn on the television any morning and you will see meteorologists talking about the days weather forecast. In addition to telling you what the weather conditions will be like (sunny, cloudy, rainy, muggy), they also tell you the days forecast for high and low temperatures. A hot summer day may reach 100° in Philadelphia, while a cool spring day may have a low of 40° in Seattle.

If you have been to other countries, though, you may notice that meteorologists measure heat and cold differently outside of the United States. For example, a TV weatherman in San Diego may forecast a high of 89°, but a similar forecaster in Tijuana, Mexico, which is only 20 miles south, may look at the same weather pattern and say that the day's high temperature is going to be 32°. Whats going on here? The difference is that the two countries use different temperature scales.

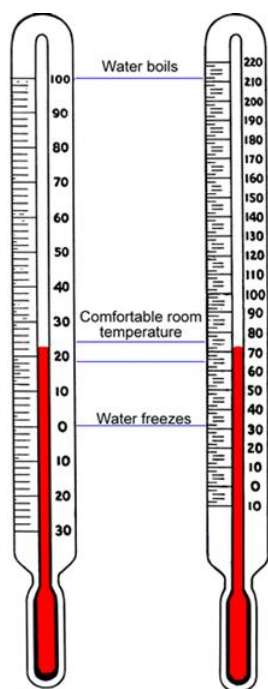
Measuring Temperature on Two Scales

Fahrenheit and Celsius are two different scales for measuring temperature.

<p>A thermometer measuring a temperature of 22° Celsius is shown here.</p> <p>On the Celsius scale, water freezes at 0° and boils at 100°.</p> <p>If the United States were to adopt the Celsius scale, forecast temperatures would rarely go below -30° or above 45°. (A temperature of</p> <p>-18° may be forecast for a cold winter day in Michigan, while a temperature of 43° may be</p>	<p>A thermometer measuring a temperature of 72° Fahrenheit is shown here.</p> <p>On the Fahrenheit scale, water freezes at 32° and boils at 212°.</p> <p>In the United States, forecast temperatures measured in Fahrenheit rarely go below -20° or above 120°. (A temperature of 0° may be forecast for a cold winter day in Michigan, while a temperature of 110° may be predicted for a hot summer day in Arizona.)</p> <p>Most office buildings maintain an indoor temperature between 65°F and 75°F to keep employees comfortable.</p>
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predicted for a hot summer day in Arizona.)

Most office buildings maintain an indoor temperature between 18°C and 24°C to keep employees comfortable.



Celsius

Fahrenheit

Try It Now

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

Converting Between the Scales

By looking at the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenheit measurement or vice versa. For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process as long as you use the formulas provided below.

Temperature Conversion Formulas

To convert a Fahrenheit measurement to a Celsius measurement, use this formula.

$$C = \frac{5}{9}(F - 32)$$

To convert a Celsius measurement to a Fahrenheit measurement, use this formula.

$$F = \frac{9}{5}C + 32$$

How were these formulas developed? They came from comparing the two scales. Since the freezing point is 0° in the Celsius scale and 32° on the Fahrenheit scale, we subtract 32 when converting from Fahrenheit to Celsius, and add 32 when converting from Celsius to Fahrenheit.

There is a reason for the fractions $\frac{5}{9}$ and $\frac{9}{5}$, also. There are 100 degrees between the freezing (0°) and boiling points (100°) of water on the Celsius scale and 180 degrees between the similar points (32° and 212°) on the Fahrenheit scale. Writing these two scales as a ratio, $\frac{F^{\circ}}{C^{\circ}}$, gives $\frac{180^{\circ}}{100^{\circ}} = \frac{180^{\circ} \div 20}{100^{\circ} \div 20} = \frac{9}{5}$. If you flip the ratio to be $\frac{C^{\circ}}{F^{\circ}}$, you get $\frac{100^{\circ}}{180^{\circ}} = \frac{100^{\circ} \div 20}{180^{\circ} \div 20} = \frac{5}{9}$. Notice how these fractions are used in the conversion formulas.

The example below illustrates the conversion of Celsius temperature to Fahrenheit temperature, using the boiling point of water, which is 100° C.

Example

The boiling point of water is 100° C. What temperature does water boil at in the Fahrenheit scale? A Celsius temperature is given. To convert it to the Fahrenheit scale, use the formula at the left.

$$F = \frac{9}{5}C + 32$$

Substitute 100 for C and multiply.

$$F = \frac{9}{5}(100) + 32$$

$$F = \frac{900}{5} + 32$$

Simplify $\frac{900}{5}$ by dividing numerator and denominator by 5.

$$F = \frac{900 \div 5}{5 \div 5} + 32$$

$$F = \frac{180}{1} + 32$$

Add $180 + 32$.

$$F = 212$$

The boiling point of water is 212° F.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

Water freezes at 32° F. On the Celsius scale, what temperature is this?

Answer

A Fahrenheit temperature is given. To convert it to the Celsius scale, use the formula at the left.

$$C = \frac{5}{9}(F - 32)$$

Substitute 32 for F and subtract.

$$C = \frac{5}{9}(32 - 32)$$

Any number multiplied by 0 is 0.

$$C = \frac{5}{9}(0)$$

$$C = 0$$

The freezing point of water is 0°C .

Try It Now

Visit this page in your course online to practice before taking the quiz.

The two previous problems used the conversion formulas to verify some temperature conversions that were discussed earlier the boiling and freezing points of water. The next example shows how these formulas can be used to solve a real-world problem using different temperature scales.

Example

Two scientists are doing an experiment designed to identify the boiling point of an unknown liquid. One scientist gets a result of 120°C ; the other gets a result of 250°F . Which temperature is higher and by how much?

Answer

One temperature is given in $^{\circ}\text{C}$, and the other is given in $^{\circ}\text{F}$. To find the difference between them, we need to measure them on the same scale.

What is the difference between 120°C and 250°F ?

Use the conversion formula to convert 120°C to $^{\circ}\text{F}$.

(You could convert 250°F to $^{\circ}\text{C}$ instead; this is explained in the text after this example.)

$$F = \frac{9}{5}C + 32$$

Substitute 120 for C .

$$F = \frac{9}{5}(120) + 32$$

Multiply.

$$F = \frac{1080}{5} + 32$$

Simplify $\frac{1080}{5}$ by dividing numerator and denominator by 5.

$$F = \frac{1080 \div 5}{5 \div 5} + 32$$

Add $216 + 32$.

$$F = \frac{216}{1} + 32$$

You have found that $120^{\circ}\text{C} = 248^{\circ}\text{F}$.

$$F = 248$$

To find the difference between 248°F and 250°F , subtract.

$$250^{\circ}\text{F} - 248^{\circ}\text{F} = 2^{\circ}\text{F}$$

250°F is the higher temperature by 2°F .

You could have converted 250°F to $^{\circ}\text{C}$ instead, and then found the difference in the two measurements. (Had you done it this way, you would have found that $250^{\circ}\text{F} = 121.1^{\circ}\text{C}$, and that 121.1°C is 1.1°C higher than 120°C .) Whichever way you choose, it is important to compare the temperature measurements within the same scale, and to apply the conversion formulas accurately.

Try It Now

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26°C. What is the average temperature in degrees Fahrenheit?

Summary

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale.

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PUTTING IT TOGETHER: MEASUREMENT

At the beginning of this module, you were lost in Paris surrounded by unfamiliar units of measurement. Now that you know a little more about measurement and how to convert among units, you can get your travel plans sorted out in no time.

Let's begin with deciding what to wear. You know the temperature on the Celsius scale is 18°, but you are much more familiar with the Fahrenheit scale. Convert Celsius to Fahrenheit in the following way:

$$F = \frac{5}{9}C + 32$$

$$F = \frac{5}{9}(18) + 32$$

$$F = 42$$

Therefore, 18°C is 42°F. You knew that the temperature was above freezing, which is 0°C, but now you have a better sense of just how cold it is based on your experiences. A warm jacket might be appropriate, and perhaps a beret since you are in France.

Now that you know what to wear, you would like to go to the top of the Eiffel Tower which is about 300 meters tall.

Remember that meters can be used to measure height or distance. Related units of measurement in the U.S. Customary system include yards, feet, and inches. You want to compare the height of the Eiffel tower and the Empire State Building, but you only know the height of the Empire state Building in feet. You can convert from meters to feet to make the comparison.

$$1m \approx 3.28ft$$

$$\frac{1m}{3.28ft} = \frac{300m}{x}$$

$$x \approx 984ft$$



The Eiffel Tower is about 984 ft tall. If the Empire State Building is 1,250 feet, it is a bit taller than the Eiffel Tower.



After touring around Paris, you decide to drive to the city of Bordeaux. You read that it is 565 km away. You are familiar with driving distances in miles, so you decide to convert kilometers to miles.

$$1mi \approx 1.61km$$

$$\frac{1mi}{1.61km} = \frac{x}{565km}$$

$$x \approx 350mi$$

If you drive at 60 miles per hour, how long will the trip take?

$$\frac{350mi}{60mph} \approx 5.83h$$

You will need to plan for about 6 hours of driving. It would not be a day trip so you will need to make plans to stay overnight.

The last thing you need to plan for is gasoline. How many liters of gasoline will you need to buy if the rental Fiat 500 uses 4.8 liters of gasoline per 100 kilometer?

$$\frac{4.8L}{100km} \times 565km = 27.12L$$

You are pleased that you will only need to fill the 35 L tank once to get to Bordeaux and once to get back.

Now you are dressed, have done some sightseeing, and you are on your way to a new adventure. Thanks to a little understanding of measurement and unit conversions, you have taken the mystery out of traveling. So go

enjoy!

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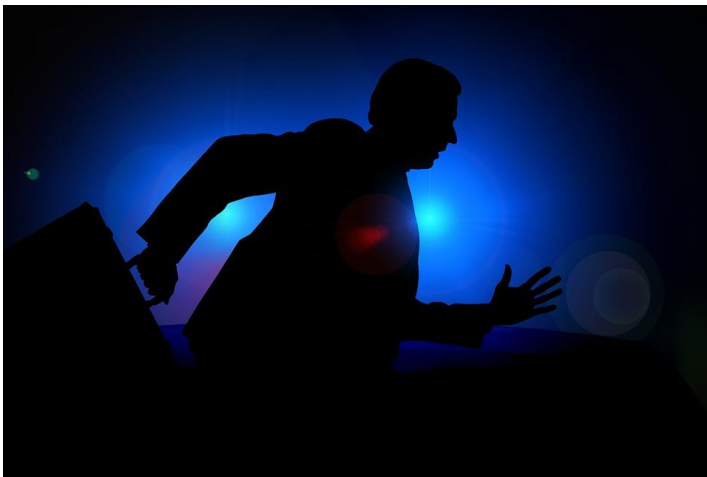
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GRAPH THEORY

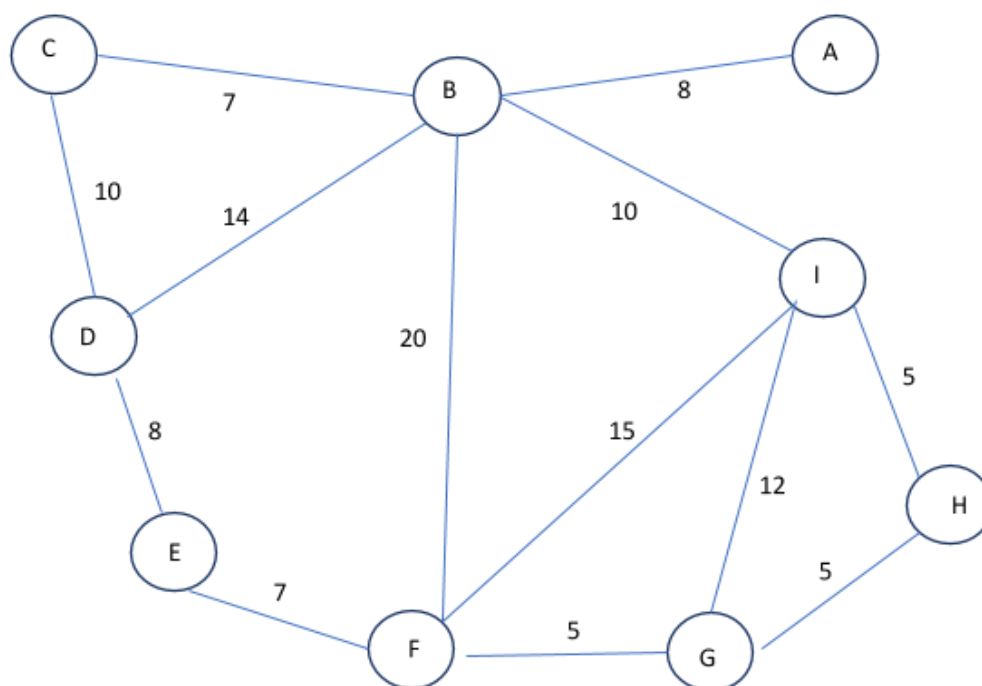
WHY IT MATTERS: GRAPH THEORY



You shined your shoes and packed your briefcase. Now it's time to stop in at each of your clients now that you are the main sales representative for your company. You have nine clients in the same region, and you have mapped them out by determining the time it will take you to get from one to the other in minutes. You labeled the clients A through I, and your original plan was to visit them in alphabetical order. Sounds easy enough. The problem is that when you finish up with client G, you realize you left a very important sample back at Client C. How can you determine the shortest path back? And then once you get it, how can you decide the fastest route to your next client?

You can start by making a diagram showing the travel time between each client. However, there are multiple paths that are possible from any one client to another. Your diagram looks a bit overwhelming – it isn't immediately obvious which route would be the fastest.

Fortunately, you'll learn how to solve the problem as you complete this module. You will figure out how to interpret diagrams such as this and use them to make logical decisions, such as which way to go to keep your clients happy and close the deal.



Learning Objectives

Elements of Graph Theory

- Define and use the elements of a graph to optimize paths through the graph
- Identify the number of vertices and edges on a graph
- Determine whether a graph is connected
- Define the degree of a vertex of a graph
- Determine the difference between a path and a circuit

Shortest Path

- Use Dijkstra's algorithm to find the shortest path between two vertices
- Given a table of driving times between cities, find the shortest path between two cities

Euler Paths

- Define an Euler path, and an Euler circuit
- Use Fleury's algorithm to determine whether a graph has an Euler circuit

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INTRODUCTION: GRAPH THEORY BASICS

Learning Objectives

In this lesson you will learn how to:

- Identify the vertices, edges, and loops of a graph
- Identify the degree of a vertex
- Identify and draw both a path and a circuit through a graph
- Determine whether a graph is connected or disconnected
- Find the shortest path through a graph using Dijkstra's Algorithm

In this lesson, we will introduce Graph Theory, a field of mathematics that started approximately 300 years ago to help solve problems such as finding the shortest path between two locations.

Now, elements of graph theory are used to optimize a wide range of systems, generate friend suggestions on social media, and plan complex shipping and air traffic routes.

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ELEMENTS OF GRAPH THEORY

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

Drawing Graphs

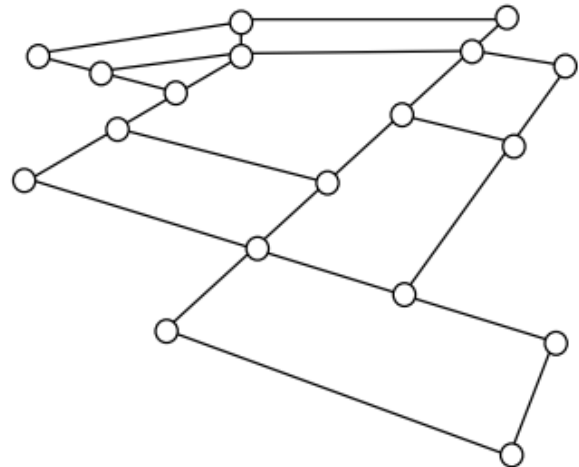
Example

Here is a portion of a housing development from Missoula, Montana. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.



Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.

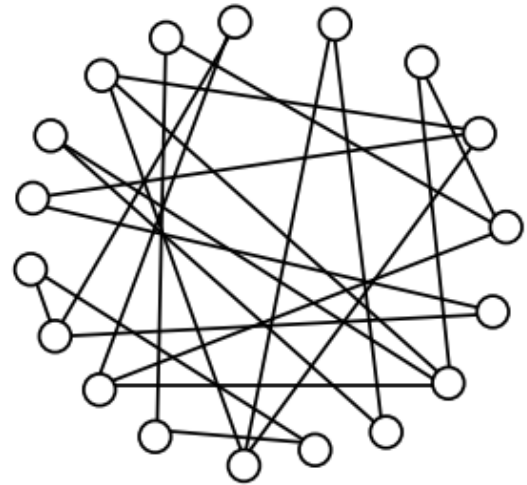
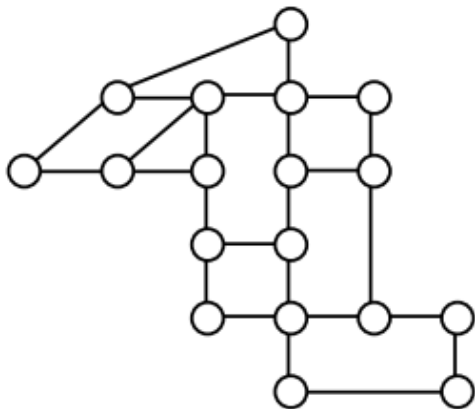


This type of simplified picture is called a **graph**.

Graphs, Vertices, and Edges

A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above.



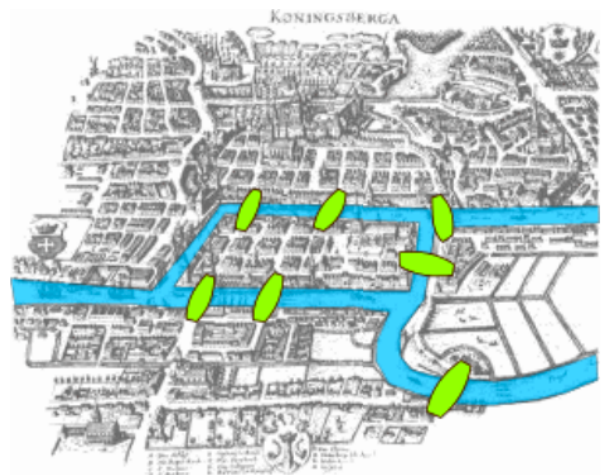
You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

Watch the video below to get another perspective of drawing a street network graph.

Watch this video online: <https://youtu.be/gyQhW24Dr2E>

Example

Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right. (Note: Bogdan Giuscă.



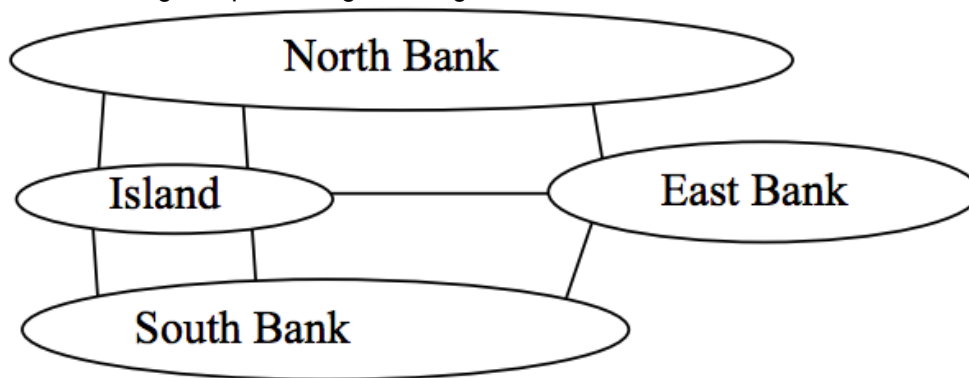
http://en.wikipedia.org/wiki/File:Königsberg_bridges.png)

As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

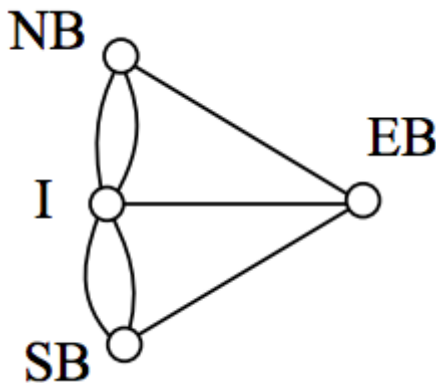
In the following video we present another view of the Königsberg bridge problem.

Watch this video online: <https://youtu.be/yn-OD8OBSDM>

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

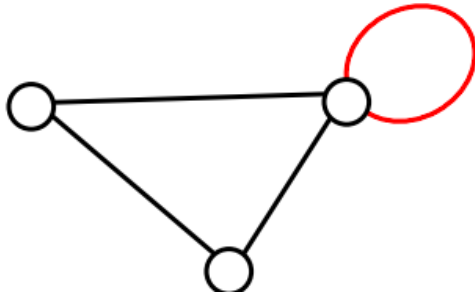
A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass—the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.



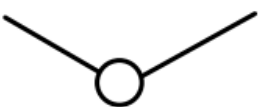
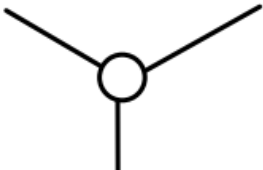
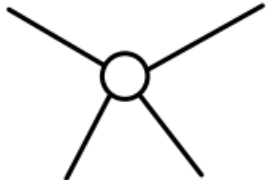
Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



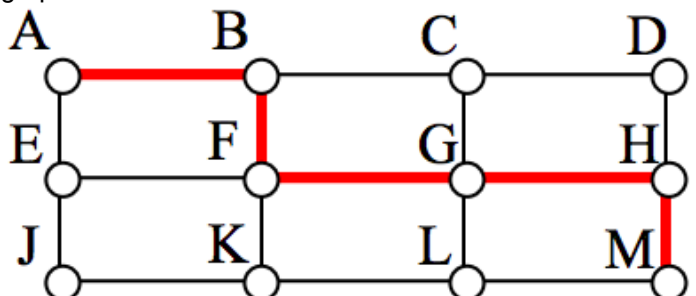
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree 2	Degree 3	Degree 4
				

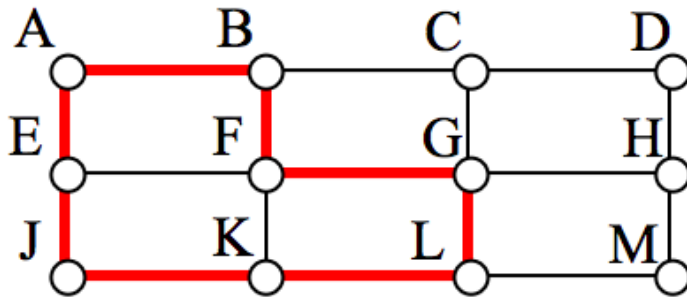
Path

A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



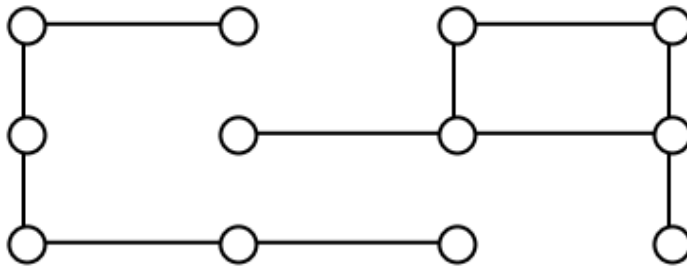
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.

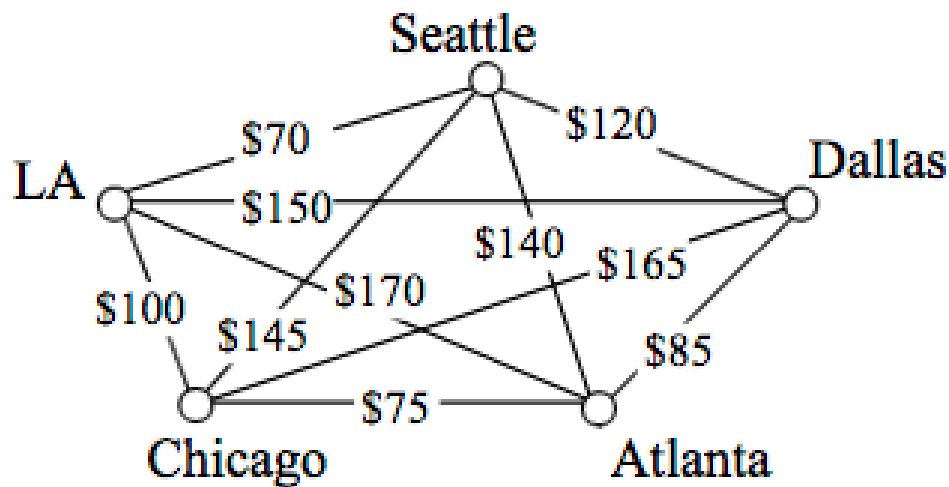


Weights

Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Try It Now

1. The graph below shows 5 cities. The weights on the edges represent the airfare for a one-way flight between the cities.
 - a. How many vertices and edges does the graph have?
 - b. Is the graph connected?
 - c. What is the degree of the vertex representing LA?
 - d. If you fly from Seattle to Dallas to Atlanta, is that a path or a circuit?
 - e. If you fly from LA to Chicago to Dallas to LA, is that a path or a circuit?



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SHORTEST PATH

When you visit a website like Google Maps or use your Smartphone to ask for directions from home to your Aunt's house in Pasadena, you are usually looking for a shortest path between the two locations. These computer applications use representations of the street maps as graphs, with estimated driving times as edge weights.

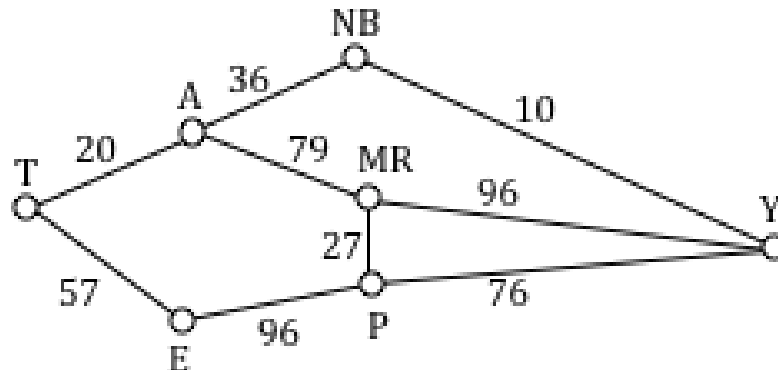
While often it is possible to find a shortest path on a small graph by guess-and-check, our goal in this chapter is to develop methods to solve complex problems in a systematic way by following **algorithms**. An algorithm is a step-by-step procedure for solving a problem. Dijkstra's (pronounced dike-stra) algorithm will find the shortest path between two vertices.

Dijkstra's Algorithm

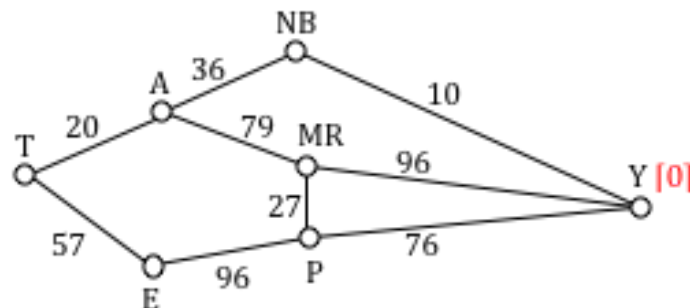
1. Mark the ending vertex with a distance of zero. Designate this vertex as current.
2. Find all vertices leading to the current vertex. Calculate their distances to the end. Since we already know the distance the current vertex is from the end, this will just require adding the most recent edge. Don't record this distance if it is longer than a previously recorded distance.
3. Mark the current vertex as visited. We will never look at this vertex again.
4. Mark the vertex with the smallest distance as current, and repeat from step 2.

EXAMPLE

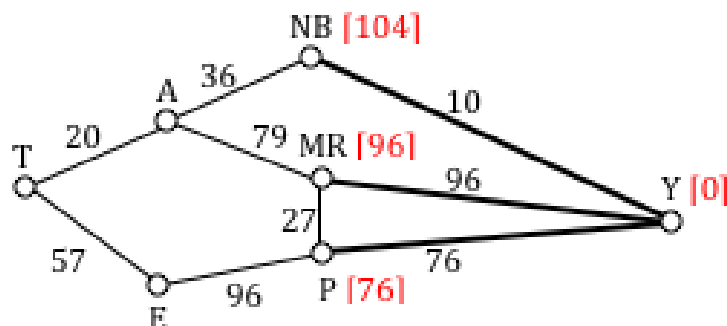
Suppose you need to travel from Tacoma, WA (vertex T) to Yakima, WA (vertex Y). Looking at a map, it looks like driving through Auburn (A) then Mount Rainier (MR) might be shortest, but it's not totally clear since that road is probably slower than taking the major highway through North Bend (NB). A graph with travel times in minutes is shown below. An alternate route through Eatonville (E) and Packwood (P) is also shown.



Step 1: Mark the ending vertex with a distance of zero. The distances will be recorded in [brackets] after the vertex name

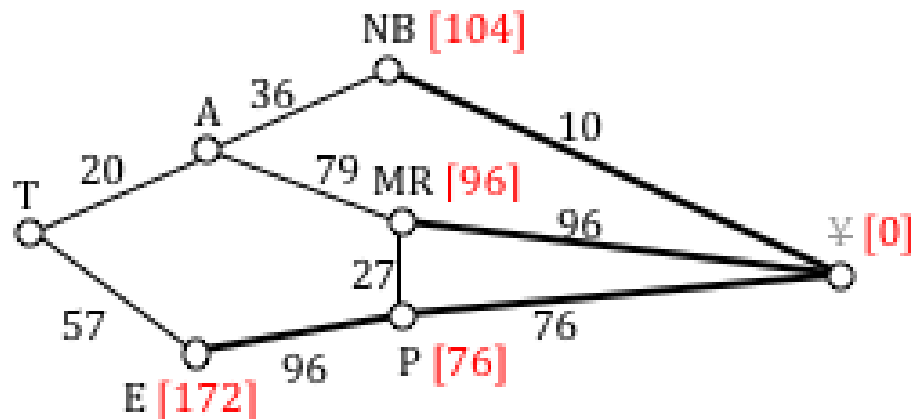


Step 2: For each vertex leading to Y, we calculate the distance to the end. For example, NB is a distance of 104 from the end, and MR is 96 from the end. Remember that distances in this case refer to the travel time in minutes.



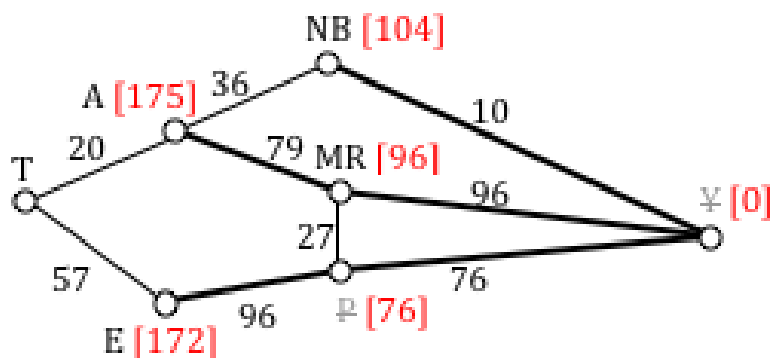
Step 3 & 4: We mark Y as visited, and mark the vertex with the smallest recorded distance as current. At this point, P will be designated current. Back to step 2.

Step 2 (#2): For each vertex leading to P (and not leading to a visited vertex) we find the distance from the end. Since E is 96 minutes from P, and we've already calculated P is 76 minutes from Y, we can compute that E is $96+76 = 172$ minutes from Y.



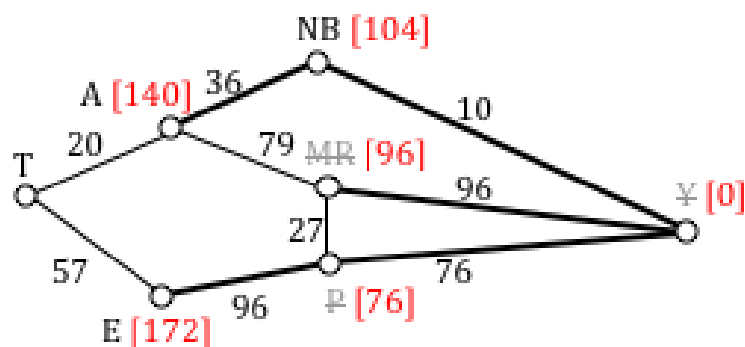
Step 3 & 4 (#2): We mark P as visited, and designate the vertex with the smallest recorded distance as current: MR. Back to step 2.

Step 2 (#3): For each vertex leading to MR (and not leading to a visited vertex) we find the distance to the end. The only vertex to be considered is A, since we've already visited Y and P. Adding MR's distance 96 to the length from A to MR gives the distance $96+79 = 175$ minutes from A to Y.



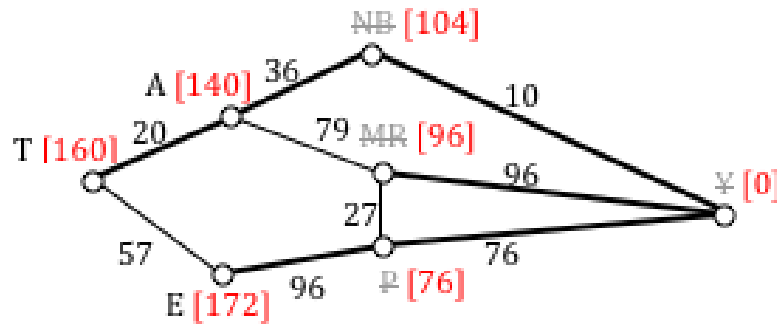
Step 3 & 4 (#3): We mark MR as visited, and designate the vertex with smallest recorded distance as current: NB. Back to step 2.

Step 2 (#4): For each vertex leading to NB, we find the distance to the end. We know the shortest distance from NB to Y is 104 and the distance from A to NB is 36, so the distance from A to Y through NB is $104+36 = 140$. Since this distance is shorter than the previously calculated distance from Y to A through MR, we replace it.



Step 3 & 4 (#4): We mark NB as visited, and designate A as current, since it now has the shortest distance.

Step 2 (#5): T is the only non-visited vertex leading to A, so we calculate the distance from T to Y through A: $20+140 = 160$ minutes.



Step 3 & 4 (#5): We mark A as visited, and designate E as current.

Step 2 (#6): The only non-visited vertex leading to E is T. Calculating the distance from T to Y through E, we compute $172 + 57 = 229$ minutes. Since this is longer than the existing marked time, we do not replace it.

Step 3 (#6): We mark E as visited. Since all vertices have been visited, we are done.

From this, we know that the shortest path from Tacoma to Yakima will take 160 minutes. Tracking which sequence of edges yielded 160 minutes, we see the shortest path is T-A-NB-Y.

Dijkstra's algorithm is an **optimal algorithm**, meaning that it always produces the actual shortest path, not just a path that is pretty short, provided one exists. This algorithm is also **efficient**, meaning that it can be implemented in a reasonable amount of time. Dijkstra's algorithm takes around V^2 calculations, where V is the number of vertices in a graph[1]. A graph with 100 vertices would take around 10,000 calculations. While that would be a lot to do by hand, it is not a lot for computer to handle. It is because of this efficiency that your car's GPS unit can compute driving directions in only a few seconds.

[1] It can be made to run faster through various optimizations to the implementation.

In contrast, an **inefficient** algorithm might try to list all possible paths then compute the length of each path. Trying to list all possible paths could easily take 1025 calculations to compute the shortest path with only 25 vertices; that's a 1 with 25 zeros after it! To put that in perspective, the fastest computer in the world would still spend over 1000 years analyzing all those paths.

EXAMPLE

A shipping company needs to route a package from Washington, D.C. to San Diego, CA. To minimize costs, the package will first be sent to their processing center in Baltimore, MD then sent as part of mass shipments between their various processing centers, ending up in their processing center in Bakersfield, CA. From there it will be delivered in a small truck to San Diego.

The travel times, in hours, between their processing centers are shown in the table below. Three hours has been added to each travel time for processing. Find the shortest path from Baltimore to Bakersfield.

	Baltimore	Denver	Dallas	Chicago	Atlanta	Bakersfield
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	

Bakersfield		19	25			*
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While we could draw a graph, we can also work directly from the table.

Step 1: The ending vertex, Bakersfield, is marked as current.

Step 2: All cities connected to Bakersfield, in this case Denver and Dallas, have their distances calculated; we'll mark those distances in the column headers.

Step 3 & 4: Mark Bakersfield as visited. Here, we are doing it by shading the corresponding row and column of the table. We mark Denver as current, shown in bold, since it is the vertex with the shortest distance.

	Baltimore	Denver [19]	Dallas [25]	Chicago	Atlanta	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#2): For cities connected to Denver, calculate distance to the end. For example, Chicago is 18 hours from Denver, and Denver is 19 hours from the end, the distance for Chicago to the end is $18+19 = 37$ (Chicago to Denver to Bakersfield). Atlanta is 24 hours from Denver, so the distance to the end is $24+19 = 43$ (Atlanta to Denver to Bakersfield).

Step 3 & 4 (#2): We mark Denver as visited and mark Dallas as current.

	Baltimore	Denver [19]	Dallas [25]	Chicago [37]	Atlanta [43]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#3): For cities connected to Dallas, calculate the distance to the end. For Chicago, the distance from Chicago to Dallas is 18 and from Dallas to the end is 25, so the distance from Chicago to the end through Dallas would be $18+25 = 43$. Since this is longer than the currently marked distance for Chicago, we do not replace it. For Atlanta, we calculate $15+25 = 40$. Since this is shorter than the currently marked distance for Atlanta, we replace the existing distance.

Step 3 & 4 (#3): We mark Dallas as visited, and mark Chicago as current.

	Baltimore	Denver [19]	Dallas [25]	Chicago [37]	Atlanta [40]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19

Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#4): Baltimore and Atlanta are the only non-visited cities connected to Chicago. For Baltimore, we calculate $15+37 = 52$ and mark that distance. For Atlanta, we calculate $14+37 = 51$. Since this is longer than the existing distance of 40 for Atlanta, we do not replace that distance.

Step 3 & 4 (#4): Mark Chicago as visited and Atlanta as current.

	Baltimore [52]	Denver [19]	Dallas [25]	Chicago [37]	Atlanta [40]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

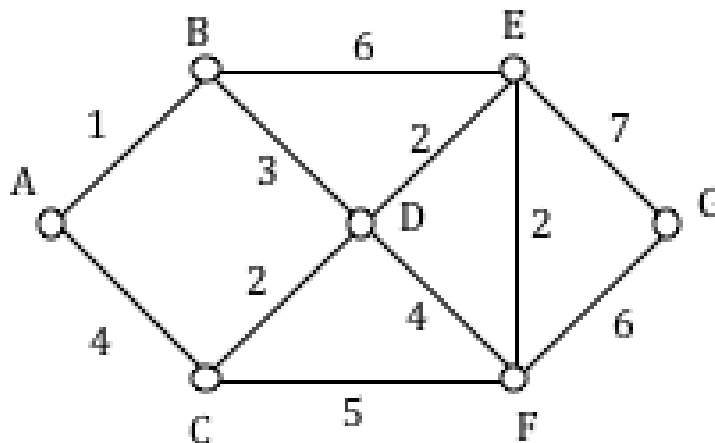
Step 2 (#5): The distance from Atlanta to Baltimore is 14. Adding that to the distance already calculated for Atlanta gives a total distance of $14+40 = 54$ hours from Baltimore to Bakersfield through Atlanta. Since this is larger than the currently calculated distance, we do not replace the distance for Baltimore.

Step 3 & 4 (#5): We mark Atlanta as visited. All cities have been visited and we are done.

The shortest route from Baltimore to Bakersfield will take 52 hours, and will route through Chicago and Denver.

Try It Now

- Find the shortest path between vertices A and G in the graph below.



Visit this page in your course online to practice before taking the quiz.
The following video summarizes the topics covered on this page.
Watch this video online: <https://youtu.be/KvRwplnIoEM>

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INTRODUCTION: EULER AND HAMILTONIAN PATHS AND CIRCUITS

Learning Objectives

- Determine whether a graph has an Euler path and/ or circuit
- Use Fleury's algorithm to find an Euler circuit
- Add edges to a graph to create an Euler circuit if one doesn't exist
- Identify whether a graph has a Hamiltonian circuit or path
- Find the optimal Hamiltonian circuit for a graph using the brute force algorithm, the nearest neighbor algorithm, and the sorted edges algorithm
- Identify a connected graph that is a spanning tree
- Use Kruskal's algorithm to form a spanning tree, and a minimum cost spanning tree

In the next lesson, we will investigate specific kinds of paths through a graph called Euler paths and circuits. Euler paths are an optimal path through a graph. They are named after him because it was Euler who first defined them.

By counting the number of vertices of a graph, and their degree we can determine whether a graph has an Euler path or circuit. We will also learn another algorithm that will allow us to find an Euler circuit once we determine that a graph has one.

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EULER CIRCUITS

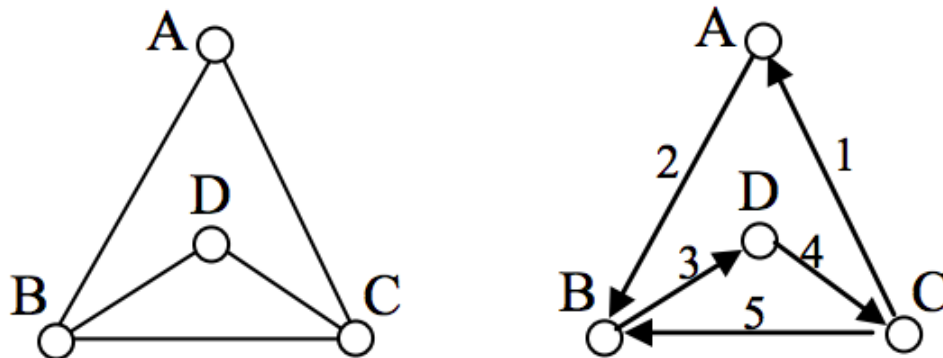
In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Path

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example

In the graph shown below, there are several Euler paths. One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

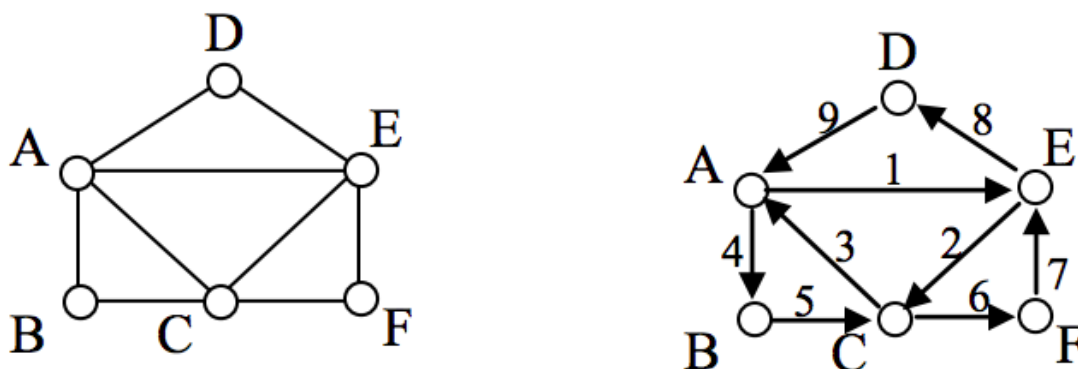


Euler Circuit

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example

The graph below has several possible Euler circuits. Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.



Look back at the example used for Euler paths—does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit *exists*.

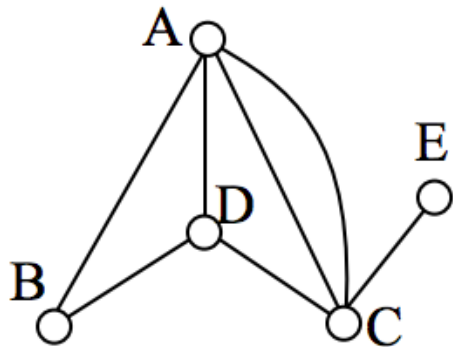
Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

Euler's Path and Circuit Theorems

A graph will contain an Euler path if it contains at most two vertices of odd degree.
A graph will contain an Euler circuit if all vertices have even degree

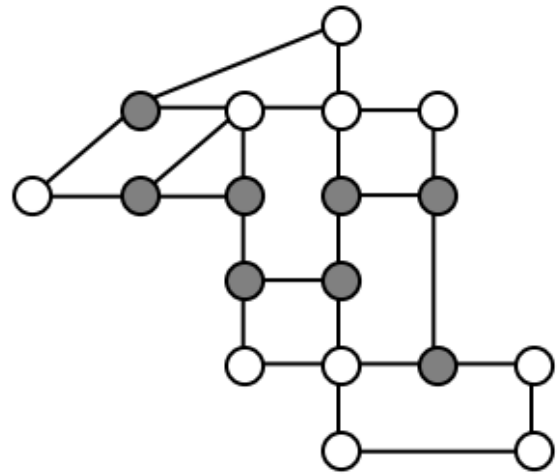
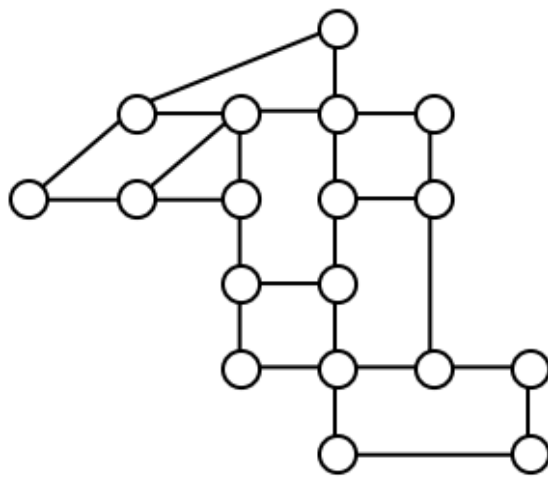
Example

In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



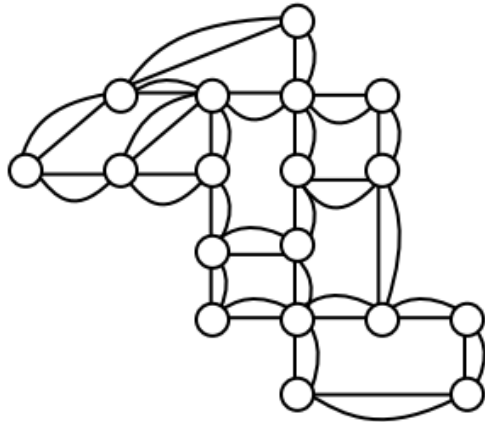
Example

Is there an Euler circuit on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.



Example

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

The following video gives more examples of how to determine an Euler path, and an Euler Circuit for a graph.

Watch this video online: <https://youtu.be/5M-m62qTR-s>

Fleury's Algorithm

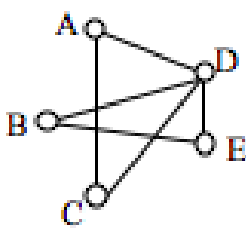
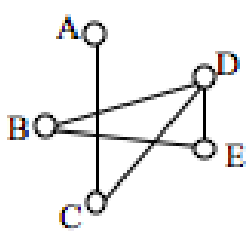
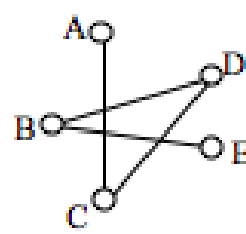
Now we know how to determine if a graph has an Euler circuit, but if it does, how do we find one? While it usually is possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is Fleury's algorithm.

Fleury's Algorithm

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

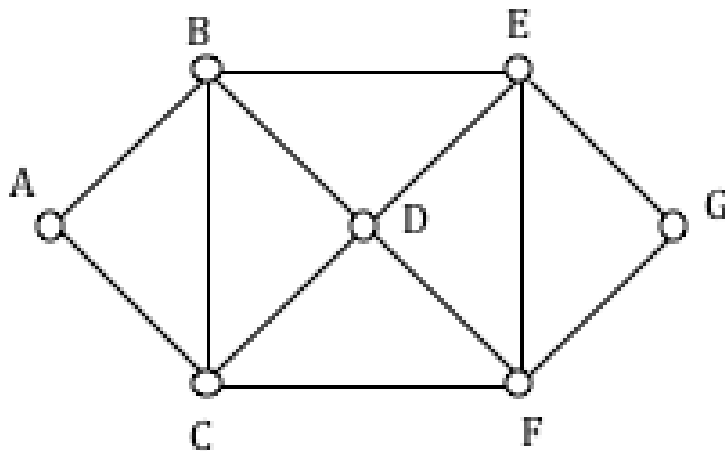
Example

Find an Euler Circuit on this graph using Fleury's algorithm, starting at vertex A.

<p>Original Graph. Choosing edge AD.</p>  <p>Circuit so far: AD</p>	<p>AD deleted. D is current. Can't choose DC since that would disconnect graph. Choosing DE</p>  <p>Circuit so far: ADE</p>	<p>E is current. From here, there is only one option, so the rest of the circuit is determined.</p>  <p>Circuit: ADEBDCA</p>
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Try It Now

Does the graph below have an Euler Circuit? If so, find one.



The following video presents more examples of using Fleury's algorithm to find an Euler Circuit.

Watch this video online: <https://youtu.be/vvP4Fg4r-Ns>

Eulerization and the Chinese Postman Problem

Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. In order to do that, she will have to duplicate some edges in the graph until an Euler circuit exists.

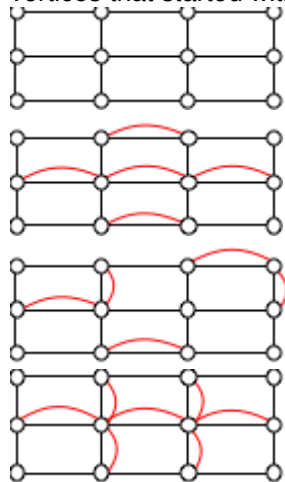
Eulerization

Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degree. Connecting two odd degree vertices increases the degree of each, giving them both even degree. When two odd degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

Note that we can only duplicate edges, not create edges where there wasn't one before. Duplicating edges would mean walking or driving down a road twice, while creating an edge where there wasn't one before is akin to installing a new road!

Example

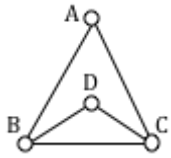
For the rectangular graph shown, three possible eulerizations are shown. Notice in each of these cases the vertices that started with odd degrees have even degrees after eulerization, allowing for an Euler circuit.



In the example above, you'll notice that the last eulerization required duplicating seven edges, while the first two only required duplicating five edges. If we were eulerizing the graph to find a walking path, we would want the eulerization with minimal duplications. If the edges had weights representing distances or costs, then we would want to select the eulerization with the minimal total added weight.

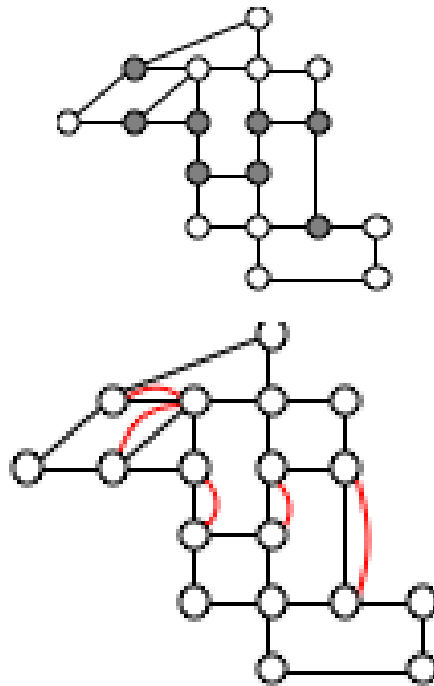
Try It Now

Eulerize the graph shown, then find an Euler circuit on the eulerized graph.



Example

Looking again at the graph for our lawn inspector from Examples 1 and 8, the vertices with odd degree are shown highlighted. With eight vertices, we will always have to duplicate at least four edges. In this case, we need to duplicate five edges since two odd degree vertices are not directly connected. Without weights we can't be certain this is the eulerization that minimizes walking distance, but it looks pretty good.



The problem of finding the optimal eulerization is called the Chinese Postman Problem, a name given by an American in honor of the Chinese mathematician Mei-Ko Kwan who first studied the problem in 1962 while trying to find optimal delivery routes for postal carriers. This problem is important in determining efficient routes for garbage trucks, school buses, parking meter checkers, street sweepers, and more.

Unfortunately, algorithms to solve this problem are fairly complex. Some simpler cases are considered in the exercises

The following video shows another view of finding an Eulerization of the lawn inspector problem.

Watch this video online: <https://youtu.be/UqCtywkskU>

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HAMILTONIAN CIRCUITS

Hamiltonian Circuits and the Traveling Salesman Problem

In the last section, we considered optimizing a walking route for a postal carrier. How is this different than the requirements of a package delivery driver? While the postal carrier needed to walk down every street (edge) to deliver the mail, the package delivery driver instead needs to visit every one of a set of delivery locations. Instead of looking for a circuit that covers every edge once, the package deliverer is interested in a circuit that visits every vertex once.

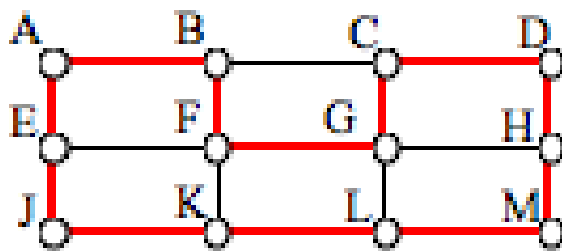
Hamiltonian Circuits and Paths

A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

Hamiltonian circuits are named for William Rowan Hamilton who studied them in the 1800's.

Example

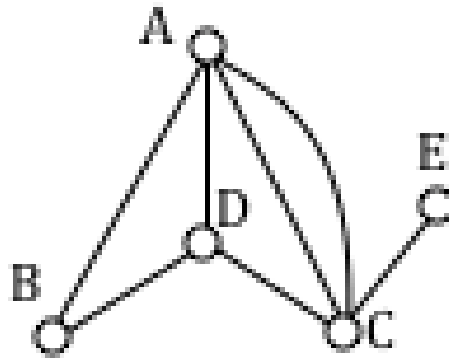
One Hamiltonian circuit is shown on the graph below. There are several other Hamiltonian circuits possible on this graph. Notice that the circuit only has to visit every vertex once; it does not need to use every edge. This circuit could be notated by the sequence of vertices visited, starting and ending at the same vertex: ABFGCDHMLKJEA. Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.



Unlike with Euler circuits, there is no nice theorem that allows us to instantly determine whether or not a Hamiltonian circuit exists for all graphs.[1]

Example

Does a Hamiltonian path or circuit exist on the graph below?



We can see that once we travel to vertex E there is no way to leave without returning to C, so there is no possibility of a Hamiltonian circuit. If we start at vertex E we can find several Hamiltonian paths, such as ECDAB and ECABD

Try It Now

Visit this page in your course online to practice before taking the quiz.

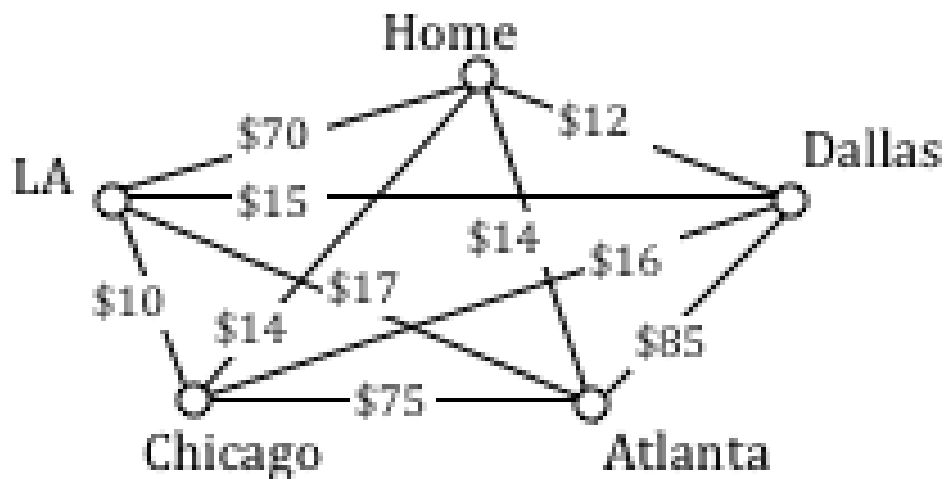
With Hamiltonian circuits, our focus will not be on existence, but on the question of optimization; given a graph where the edges have weights, can we find the optimal Hamiltonian circuit; the one with lowest total weight.

Watch this video to see the examples above worked out.

Watch this video online: <https://youtu.be/SjtVuw4-1Qo>

This problem is called the **Traveling salesman problem** (TSP) because the question can be framed like this: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city, and puts the costs in a graph. In what order should he travel to visit each city once then return home with the lowest cost?

To answer this question of how to find the lowest cost Hamiltonian circuit, we will consider some possible approaches. The first option that might come to mind is to just try all different possible circuits.



question can be framed like this: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city, and puts the costs in a graph. In what order should he travel to visit each city once then return home with the lowest cost?

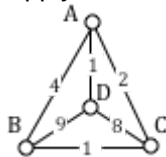
To answer this question of how to find the lowest cost Hamiltonian circuit, we will consider some possible approaches. The first option that might come to mind is to just try all different possible circuits.

Brute Force Algorithm (a.k.a. exhaustive search)

1. List all possible Hamiltonian circuits
2. Find the length of each circuit by adding the edge weights
3. Select the circuit with minimal total weight.

Example

Apply the Brute force algorithm to find the minimum cost Hamiltonian circuit on the graph below.



To apply the Brute force algorithm, we list all possible Hamiltonian circuits and calculate their weight:

Circuit	Weight
ABCD	$4+13+8+1 = 26$
ABDC	$4+9+8+2 = 23$
ACBD	$2+13+9+1 = 25$

Note: These are the unique circuits on this graph. All other possible circuits are the reverse of the listed ones or start at a different vertex, but result in the same weights.

From this we can see that the second circuit, ABDC, is the optimal circuit.

Watch these examples worked again in the following video.

Watch this video online: <https://youtu.be/wDXQ6tWsJxw>

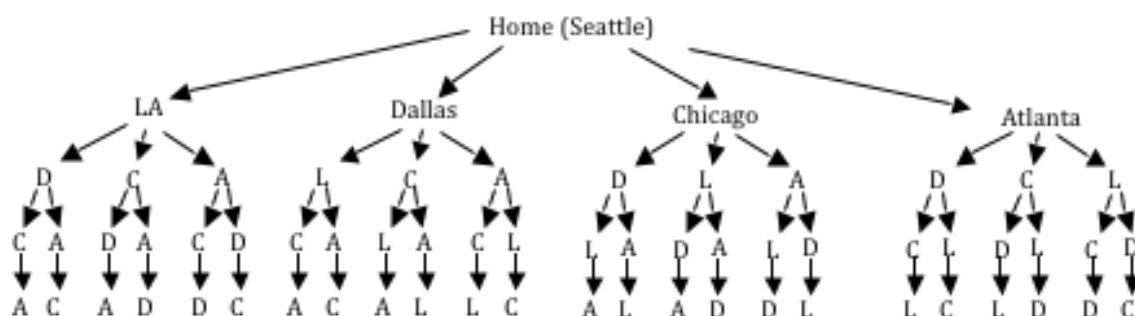
Try It Now

Visit this page in your course online to practice before taking the quiz.

The Brute force algorithm is optimal; it will always produce the Hamiltonian circuit with minimum weight. Is it efficient? To answer that question, we need to consider how many Hamiltonian circuits a graph could have. For simplicity, let's look at the worst-case possibility, where every vertex is connected to every other vertex. This is called a **complete graph**.

Suppose we had a complete graph with five vertices like the air travel graph above. From Seattle there are four cities we can visit first. From each of those, there are three choices. From each of those cities, there are two possible cities to visit next. There is then only one choice for the last city before returning home.

This can be shown visually:



Counting the number of routes, we can see there are $4 \cdot 3 \cdot 2 \cdot 1$ routes. For six cities there would be $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ routes.

Number of Possible Circuits

For N vertices in a complete graph, there will be $(n - 1)! = (n - 1)(n - 2)(n - 3) \dots 3 \cdot 2 \cdot 1$ routes. Half of these are duplicates in reverse order, so there are $\frac{(n-1)!}{2}$ unique circuits.

The exclamation symbol, $!$, is read “factorial” and is shorthand for the product shown.

Example

How many circuits would a complete graph with 8 vertices have?

A complete graph with 8 vertices would have = 5040 possible Hamiltonian circuits. Half of the circuits are duplicates of other circuits but in reverse order, leaving 2520 unique routes.

While this is a lot, it doesn't seem unreasonably huge. But consider what happens as the number of cities increase:

Cities	Unique Hamiltonian Circuits
9	$8!/2 = 20,160$
10	$9!/2 = 181,440$
11	$10!/2 = 1,814,400$
15	$14!/2 = 43,589,145,600$
20	$19!/2 = 60,822,550,204,416,000$

Watch these examples worked again in the following video.

Watch this video online: <https://youtu.be/DwZw4t0qxuQ>

As you can see the number of circuits is growing extremely quickly. If a computer looked at one billion circuits a second, it would still take almost two years to examine all the possible circuits with only 20 cities! Certainly Brute Force is **not** an efficient algorithm.

Nearest Neighbor Algorithm (NNA)

1. Select a starting point.
2. Move to the nearest unvisited vertex (the edge with smallest weight).
3. Repeat until the circuit is complete.

Unfortunately, no one has yet found an efficient *and* optimal algorithm to solve the TSP, and it is very unlikely anyone ever will. Since it is not practical to use brute force to solve the problem, we turn instead to **heuristic algorithms**; efficient algorithms that give approximate solutions. In other words, heuristic algorithms are fast, but may or may not produce the optimal circuit.

Example

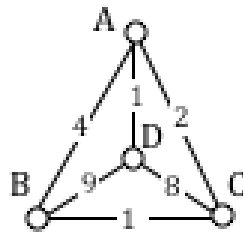
Consider our earlier graph, shown to the right.

Starting at vertex A, the nearest neighbor is vertex D with a weight of 1.

From D, the nearest neighbor is C, with a weight of 8.

From C, our only option is to move to vertex B, the only unvisited vertex, with a cost of 13.

From B we return to A with a weight of 4.



The resulting circuit is ADCBA with a total weight of $1 + 8 + 13 + 4 = 26$.

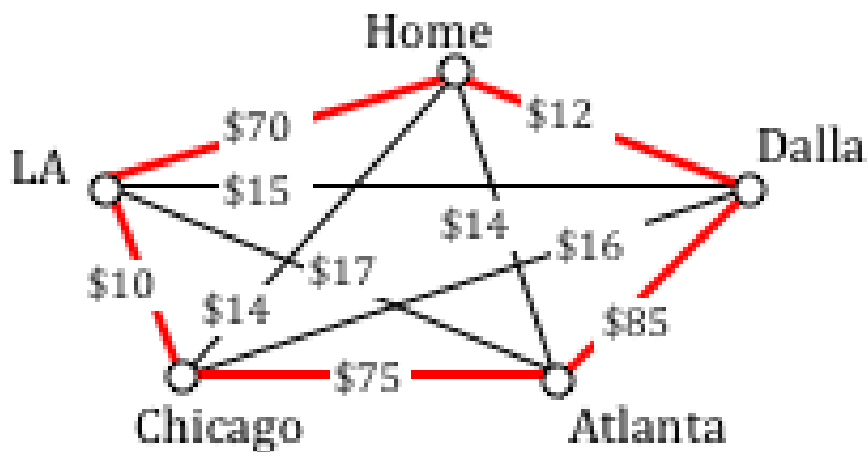
Watch the example worked out in the following video.

Watch this video online: <https://youtu.be/SqOP5n9bNX4>

We ended up finding the worst circuit in the graph! What happened? Unfortunately, while it is very easy to implement, the NNA is a **greedy** algorithm, meaning it only looks at the immediate decision without considering the consequences in the future. In this case, following the edge AD forced us to use the very expensive edge BC later.

Example

Consider again our salesman. Starting in Seattle, the nearest neighbor (cheapest flight) is to LA, at a cost of \$70. From there:



LA to Chicago: \$100
 Chicago to Atlanta: \$75
 Atlanta to Dallas: \$85
 Dallas to Seattle: \$120
 Total cost: \$450

In this case, nearest neighbor did find the optimal circuit.

Watch this example worked out again in this video.

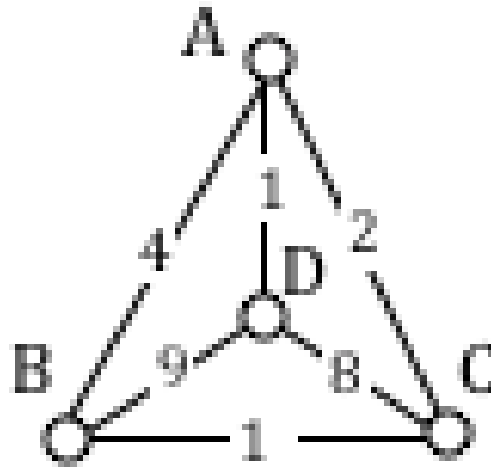
Watch this video online: <https://youtu.be/3Eq36iqjGKI>

Going back to our first example, how could we improve the outcome? One option would be to redo the nearest neighbor algorithm with a different starting point to see if the result changed. Since nearest neighbor is so fast,

doing it several times isn't a big deal.

Example

We will revisit the graph from Example 17.



Starting at vertex A resulted in a circuit with weight 26.

Starting at vertex B, the nearest neighbor circuit is BADCB with a weight of $4+1+8+13 = 26$. This is the same circuit we found starting at vertex A. No better.

Starting at vertex C, the nearest neighbor circuit is CADBC with a weight of $2+1+9+13 = 25$. Better!

Starting at vertex D, the nearest neighbor circuit is DACBA. Notice that this is actually the same circuit we found starting at C, just written with a different starting vertex.

The RNNA was able to produce a slightly better circuit with a weight of 25, but still not the optimal circuit in this case. Notice that even though we found the circuit by starting at vertex C, we could still write the circuit starting at A: ADBCA or ACBDA.

Try It Now

The table below shows the time, in milliseconds, it takes to send a packet of data between computers on a network. If data needed to be sent in sequence to each computer, then notification needed to come back to the original computer, we would be solving the TSP. The computers are labeled A-F for convenience.

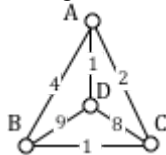
	A	B	C	D	E	F
A	—	44	34	12	40	41
B	44	—	31	43	24	50
C	34	31	—	20	39	27
D	12	43	20	—	11	17
E	40	24	39	11	—	42
F	41	50	27	17	42	—

- Find the circuit generated by the NNA starting at vertex B.
- Find the circuit generated by the RNNA.

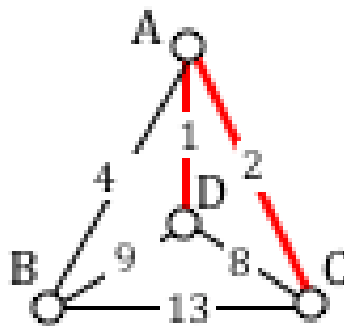
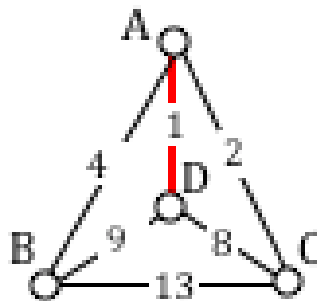
While certainly better than the basic NNA, unfortunately, the RNNA is still greedy and will produce very bad results for some graphs. As an alternative, our next approach will step back and look at the “big picture” – it will select first the edges that are shortest, and then fill in the gaps.

Example

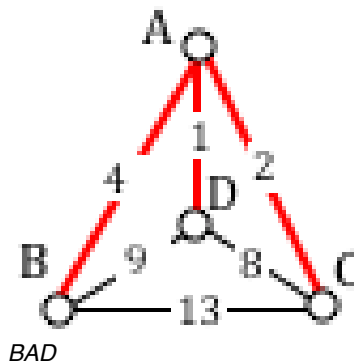
Using the four vertex graph from earlier, we can use the Sorted Edges algorithm.

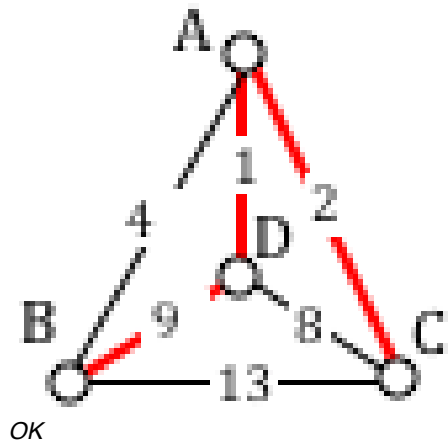
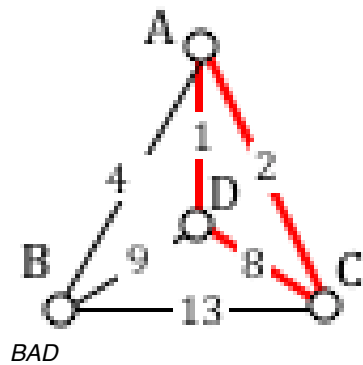


The cheapest edge is AD, with a cost of 1. We highlight that edge to mark it selected. The next shortest edge is AC, with a weight of 2, so we highlight that edge.

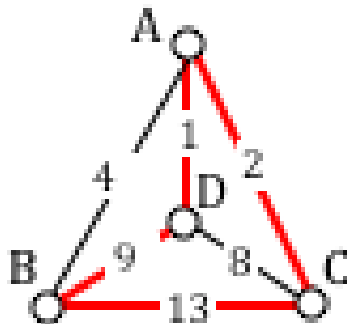


For the third edge, we'd like to add AB, but that would give vertex A degree 3, which is not allowed in a Hamiltonian circuit. The next shortest edge is CD, but that edge would create a circuit ACDA that does not include vertex B, so we reject that edge. The next shortest edge is BD, so we add that edge to the graph.





We then add the last edge to complete the circuit: ACBDA with weight 25. Notice that the algorithm did not produce the optimal circuit in this case; the optimal circuit is ACDBA with weight 23.



While the Sorted Edge algorithm overcomes some of the shortcomings of NNA, it is still only a heuristic algorithm, and does not guarantee the optimal circuit.

Example

Your teacher's band, *Derivative Work*, is doing a bar tour in Oregon. The driving distances are shown below. Plan an efficient route for your teacher to visit all the cities and return to the starting location. Use NNA starting at Portland, and then use Sorted Edges.

	Ashland	Astoria	Bend	Corvallis	Crater Lake	Eugene	Newport	Portland	Salen

Ashland	–	374	200	223	108	178	252	285	240
Astoria	374	–	255	166	433	199	135	95	136
Bend	200	255	–	128	277	128	180	160	131
Corvallis	223	166	128	–	430	47	52	84	40
Crater Lake	108	433	277	430	–	453	478	344	389
Eugene	178	199	128	47	453	–	91	110	64
Newport	252	135	180	52	478	91	–	114	83
Portland	285	95	160	84	344	110	114	–	47
Salem	240	136	131	40	389	64	83	47	–
Seaside	356	17	247	155	423	181	117	78	118

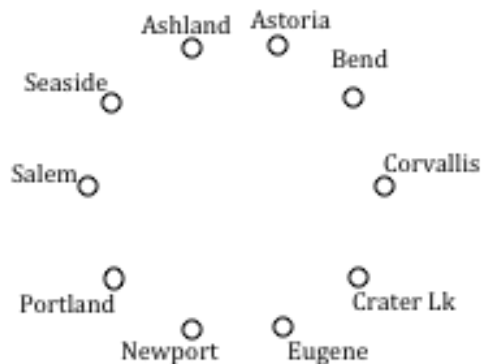
To see the entire table, scroll to the right

Using NNA with a large number of cities, you might find it helpful to mark off the cities as they're visited to keep from accidentally visiting them again. Looking in the row for Portland, the smallest distance is 47, to Salem. Following that idea, our circuit will be:

Portland to Salem 47
Salem to Corvallis 40
Corvallis to Eugene 47
Eugene to Newport 91
Newport to Seaside 117
Seaside to Astoria 17
Astoria to Bend 255
Bend to Ashland 200
Ashland to Crater Lake 108
Crater Lake to Portland 344

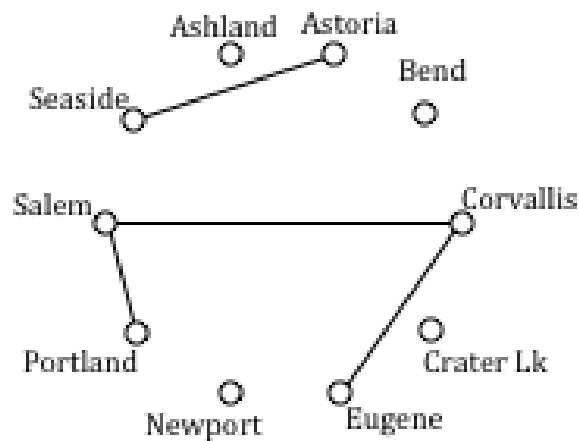
Total trip length: 1266 miles

Using Sorted Edges, you might find it helpful to draw an empty graph, perhaps by drawing vertices in a circular pattern. Adding edges to the graph as you select them will help you visualize any circuits or vertices with degree 3.



We start adding the shortest edges:

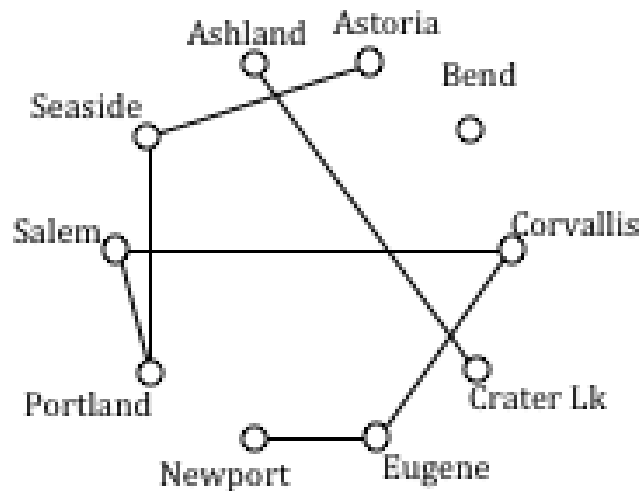
Seaside to Astoria 17 miles
Corvallis to Salem 40 miles
Portland to Salem 47 miles
Corvallis to Eugene 47 miles



The graph after adding these edges is shown to the right. The next shortest edge is from Corvallis to Newport at 52 miles, but adding that edge would give Corvallis degree 3.

Continuing on, we can skip over any edge pair that contains Salem or Corvallis, since they both already have degree 2.

Portland to Seaside	78 miles
Eugene to Newport	91 miles
Portland to Astoria	(reject – closes circuit)
Ashland to Crater Lk	108 miles



The graph after adding these edges is shown to the right. At this point, we can skip over any edge pair that contains Salem, Seaside, Eugene, Portland, or Corvallis since they already have degree 2.

Newport to Astoria	(reject – closes circuit)
Newport to Bend	180 miles
Bend to Ashland	200 miles

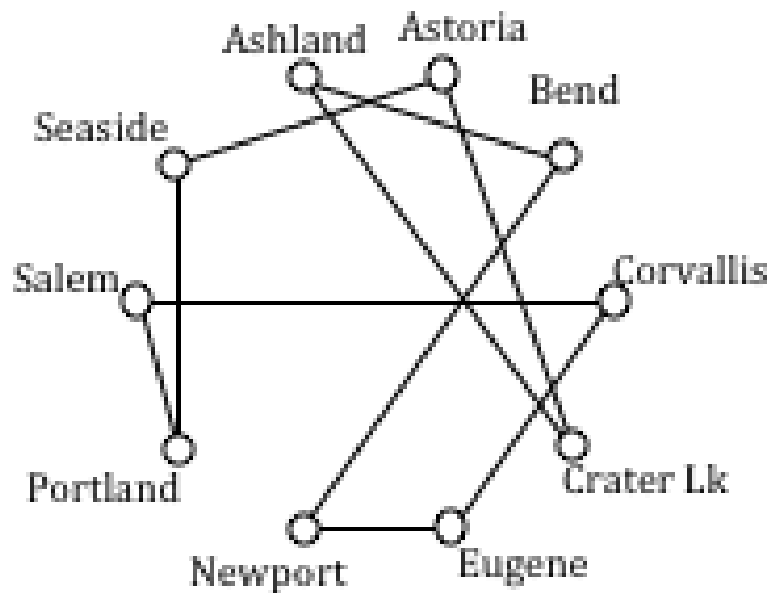
At this point the only way to complete the circuit is to add:

Crater Lk to Astoria 433 miles. The final circuit, written to start at Portland, is:

Portland, Salem, Corvallis, Eugene, Newport, Bend, Ashland, Crater Lake, Astoria, Seaside, Portland.
Total trip length: 1241 miles.

While better than the NNA route, neither algorithm produced the optimal route. The following route can make the tour in 1069 miles:

Portland, Astoria, Seaside, Newport, Corvallis, Eugene, Ashland, Crater Lake, Bend, Salem, Portland



Watch the example of nearest neighbor algorithm for traveling from city to city using a table worked out in the video below.

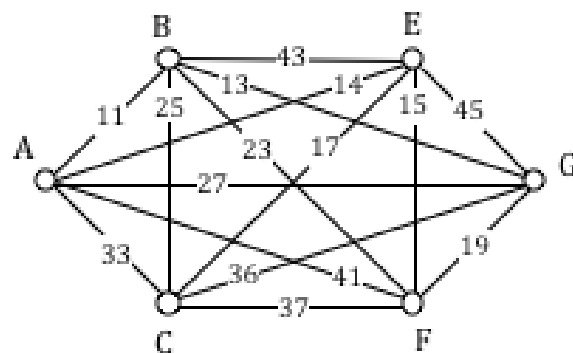
Watch this video online: <https://youtu.be/GFp-046PQx0>

In the next video we use the same table, but use sorted edges to plan the trip.

Watch this video online: https://youtu.be/_gXyujMsrnw

Try It Now

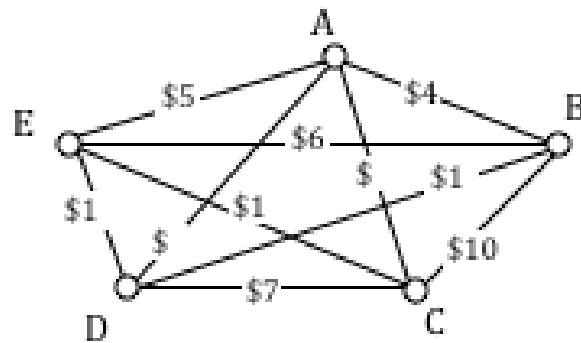
Find the circuit produced by the Sorted Edges algorithm using the graph below.



Visit this page in your course online to practice before taking the quiz.

Spanning Trees

A company requires reliable internet and phone connectivity between their five offices (named A, B, C, D, and E for simplicity) in New York, so they decide to lease dedicated lines from the phone company. The phone company will charge for each link made. The costs, in thousands of dollars per year, are shown in the graph.



In this case, we don't need to find a circuit, or even a specific path; all we need to do is make sure we can make a call from any office to any other. In other words, we need to be sure there is a path from any vertex to any other vertex.

Spanning Tree

A **spanning tree** is a connected graph using all vertices in which there are no circuits. In other words, there is a path from any vertex to any other vertex, but no circuits.

Some examples of spanning trees are shown below. Notice there are no circuits in the trees, and it is fine to have vertices with degree higher than two.



Usually we have a starting graph to work from, like in the phone example above. In this case, we form our spanning tree by finding a **subgraph** – a new graph formed using all the vertices but only some of the edges from the original graph. No edges will be created where they didn't already exist.

Of course, any random spanning tree isn't really what we want. We want the **minimum cost spanning tree (MCST)**.

Minimum Cost Spanning Tree (MCST)

The minimum cost spanning tree is the spanning tree with the smallest total edge weight.

A nearest neighbor style approach doesn't make as much sense here since we don't need a circuit, so instead we will take an approach similar to sorted edges.

Kruskal's Algorithm

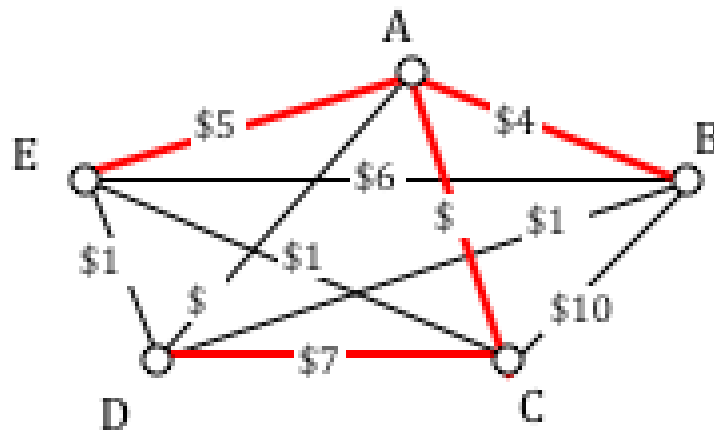
1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge, unless:
3. adding the edge would create a circuit

Repeat until a spanning tree is formed

Example

Using our phone line graph from above, begin adding edges:

AB	\$4	OK
AE	\$5	OK
BE	\$6	reject – closes circuit ABEA
DC	\$7	OK
AC	\$8	OK



At this point we stop – every vertex is now connected, so we have formed a spanning tree with cost \$24 thousand a year.

Remarkably, Kruskal's algorithm is both optimal and efficient; we are guaranteed to always produce the optimal MCST.

Example

The power company needs to lay updated distribution lines connecting the ten Oregon cities below to the power grid. How can they minimize the amount of new line to lay?

	Ashland	Astoria	Bend	Corvallis	Crater Lake	Eugene	Newport	Portland	Salen
Ashland	–	374	200	223	108	178	252	285	240
Astoria	374	–	255	166	433	199	135	95	136
Bend	200	255	–	128	277	128	180	160	131
Corvallis	223	166	128	–	430	47	52	84	40
Crater Lake	108	433	277	430	–	453	478	344	389
Eugene	178	199	128	47	453	–	91	110	64

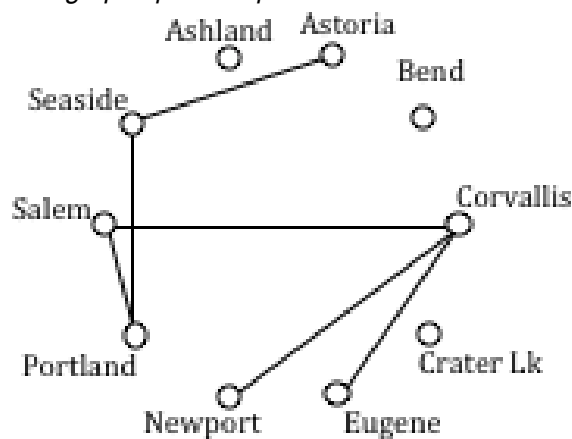
Newport	252	135	180	52	478	91	–	114	83
Portland	285	95	160	84	344	110	114	–	47
Salem	240	136	131	40	389	64	83	47	–
Seaside	356	17	247	155	423	181	117	78	118

To see the entire table, scroll to the right

Using Kruskal's algorithm, we add edges from cheapest to most expensive, rejecting any that close a circuit. We stop when the graph is connected.

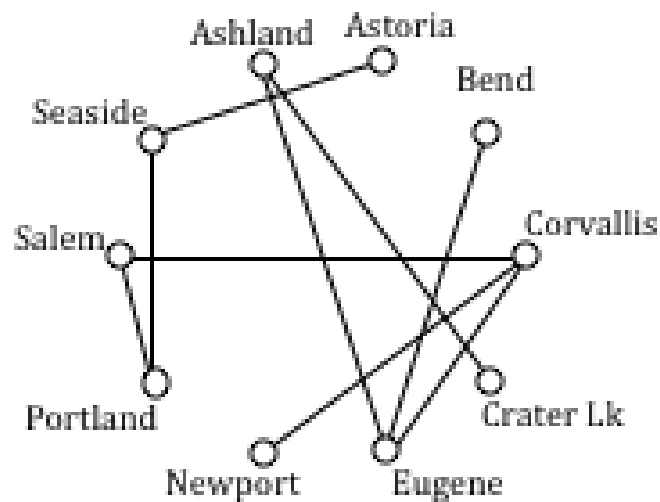
Seaside to Astoria 17 miles
Portland to Salem 47 miles
Corvallis to Eugene 47 miles
Corvallis to Newport 52 miles
Salem to Eugene reject – closes circuit
Portland to Seaside 78 miles

The graph up to this point is shown below.



Continuing,
Newport to Salem reject
Corvallis to Portland reject
Eugene to Newport reject
Portland to Astoria reject
Ashland to Crater Lk 108 miles
Eugene to Portland reject
Newport to Portland reject
Newport to Seaside reject
Salem to Seaside reject
Bend to Eugene 128 miles
Bend to Salem reject
Astoria to Newport reject
Salem to Astoria reject
Corvallis to Seaside reject
Portland to Bend reject
Astoria to Corvallis reject
Eugene to Ashland 178 miles

This connects the graph. The total length of cable to lay would be 695 miles.



Watch the example above worked out in the following video, without a table.

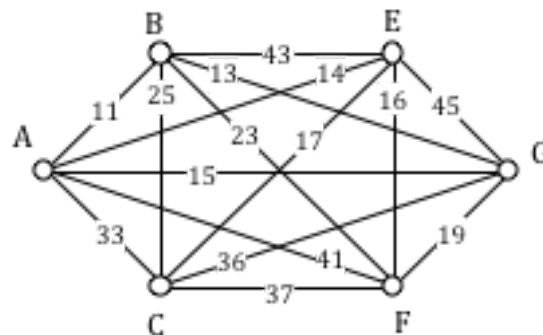
Watch this video online: <https://youtu.be/gaXM0HNErc4>

Now we present the same example, with a table in the following video.

Watch this video online: https://youtu.be/Pu2_2ftkwdo

Try It Now

Find a minimum cost spanning tree on the graph below using Kruskal's algorithm.



Visit this page in your course online to practice before taking the quiz.

[1] There are some theorems that can be used in specific circumstances, such as Dirac's theorem, which says that a Hamiltonian circuit must exist on a graph with n vertices if each vertex has degree $n/2$ or greater.

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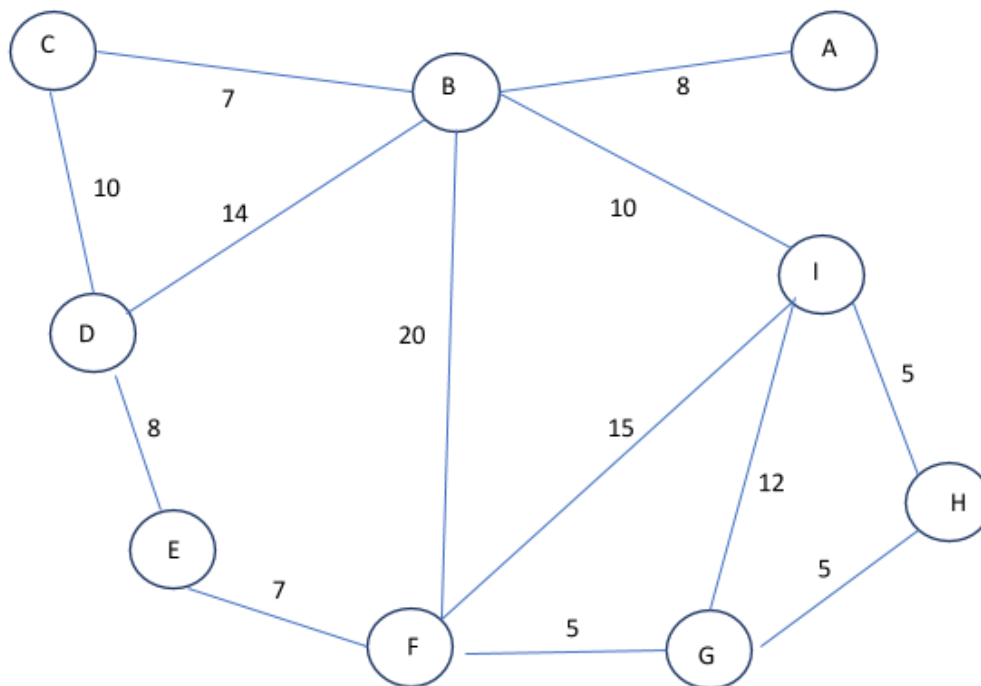
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PUTTING IT TOGETHER: GRAPH THEORY

At the start of this module, you were thinking about finding the fastest route from one client to another.

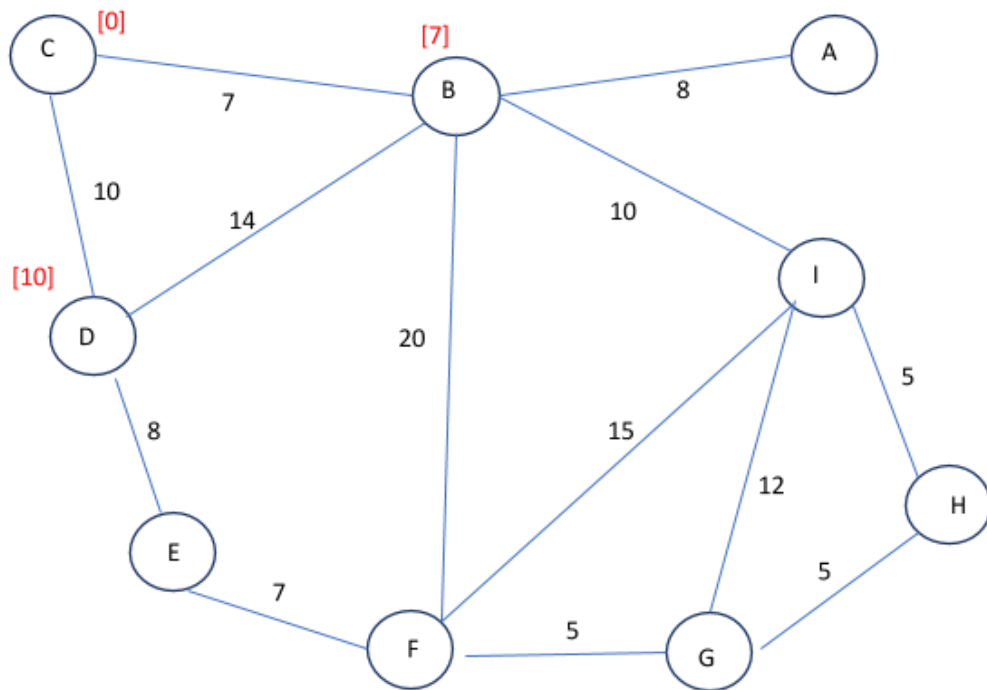
Thankfully, now you know how to use graph theory to figure it out. Take another look at the diagram. Now you know it is a graph in which each vertex represents a client and each edge represents a path between clients.

The numbers, or weights, indicate the length of time (minutes) required to travel from one vertex to another.

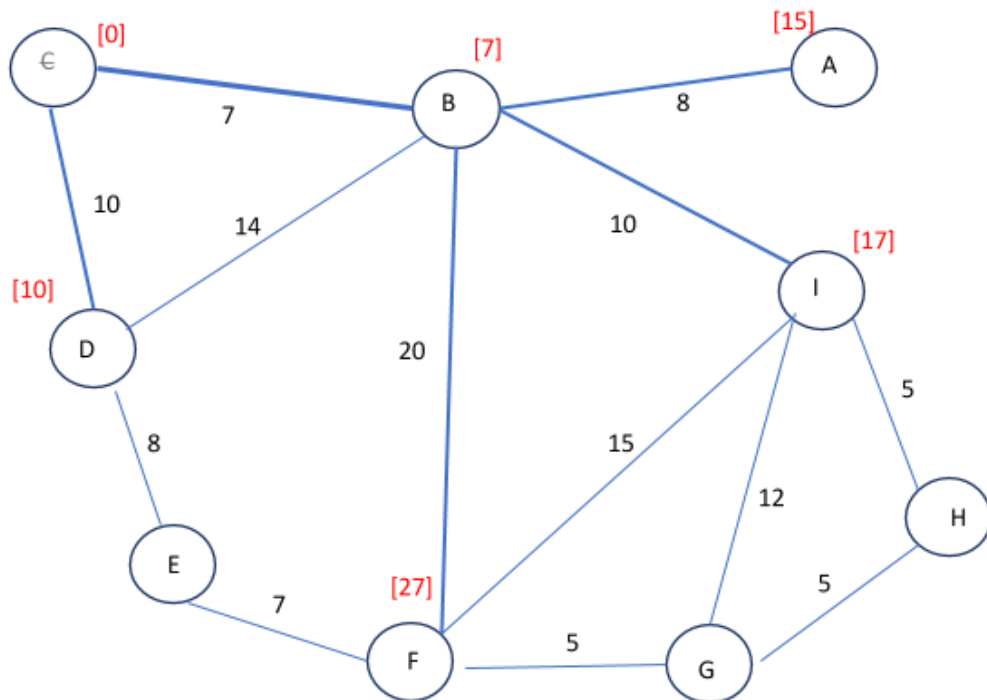


One vertex, A, is degree 1. Some vertices are degree 2, such as C and E, and H. Others are degree 3, such as D and G. Still others are degree 4, such as B, F, and I. What does that mean for you? It means that there are multiple paths leading from each client and several different routes you can take traveling from one to the other.

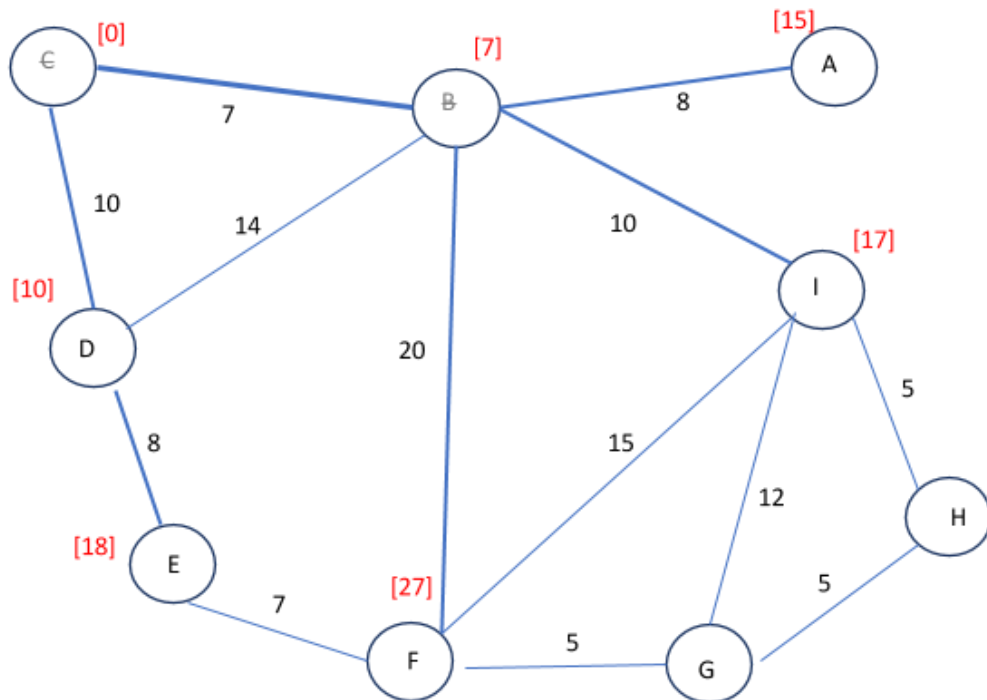
You were faced with the problem of having to get from Client G to Client C using the fastest route. Start by marking the ending vertex with a time of 0. Then find all vertices leading to the current vertex. Calculate their distances to the end.



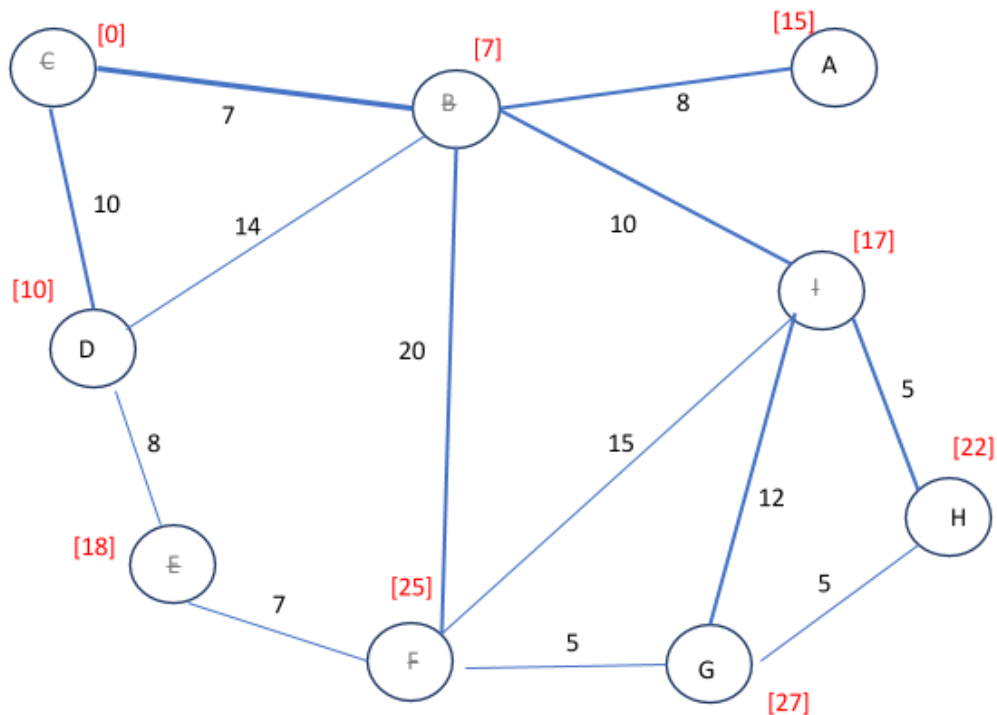
Mark C as visited, and mark the vertex with the smallest recorded time as current. So B will be designated current. Again find all vertices leading to the current vertex. For each vertex leading to B, and not to a visited vertex, find the time from the end.



Mark B as visited, and mark the vertex with the smallest recorded time as current. So D will be designated current. Add CD to DE, $10 + 8 = 18$.



Compare the time at E with that at I. The time at I is lower so mark E as visited, and mark I as current. So C to G through I has a total of 32 minutes. But the time at H is only 22 minutes. Since it is lower, keep going. Mark I as visited and H as current. The total time from C to G through H is 27 minutes. This is the fastest route.



So it will take you 27 minutes to get back to Client C to pick up your sample. It may have looked as if a different route, perhaps D, would have been faster but it turns out not to be the case. Using the graph confirmed the best route.

Now you have to get all the way to your next client at H. You can tell by the work you have already done that the fast route back to H is from C to B to I to H. And how long will that take? It will take a speedy 22 minutes. Drawing a graph may seem complex, but in the end it really helps you make definite calculations rather than estimating. And that can mean the difference between closing the deal or missing the client altogether.

FRACTALS

WHY IT MATTERS: FRACTALS

What are fractals?

Fractals are everywhere! If you don't believe me, just take a look outside your window. From the shapes of trees and bushes to the jagged profiles of mountains to the irregular coastlines, many features of our natural world seem to be modeled by fractal geometry.

But what exactly is a fractal? As you will learn in this module, a **fractal** is an object that displays self-similarity at every level. That is, when you zoom in on one section, it resembles the whole image. This self-similarity doesn't have to be exact; in fact many fractals show some variation or randomness. Below is a video illustrating how the Mandelbrot set, a well-known fractal, displays self-similarity.

Watch this video online: https://youtu.be/G_GBwuYuOOs

While some fractals (like the Mandelbrot set) could pass for works of art, the true beauty of fractals is in how such intricate designs and patterns can result from very elementary generating formulas or rules.

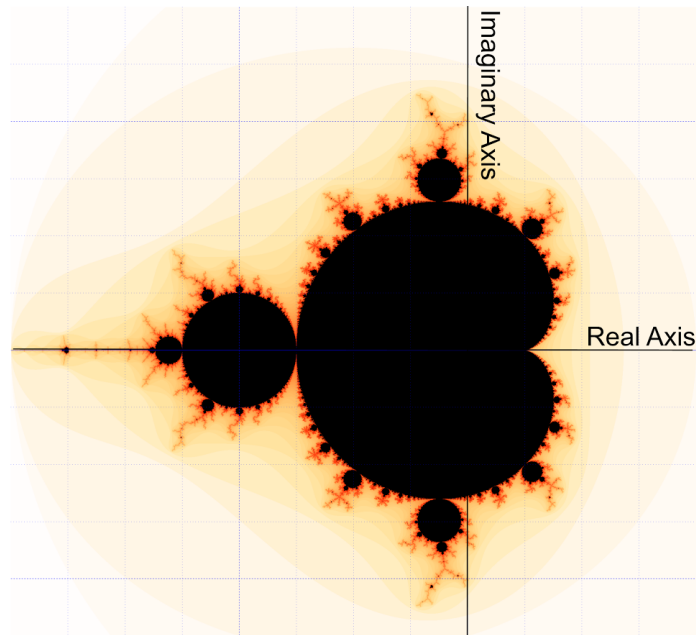
In this module, you will learn how to create fractal patterns such as the Mandelbrot set using a simple formula such as:

$$z_{n+1} = z_n^2 + c$$

Of course there are many details that still need to be explained, such as the relationship between fractals and complex numbers. The values of c , z_n and z_{n+1} in the above formula are supposed to be **complex numbers**, that is, numbers that include the **imaginary unit**, $i = \sqrt{-1}$.

The imaginary number i is something completely different than any number you have ever seen. In fact, i does not show up on the number line at all! Instead, as you will soon discover, the imaginary unit lives on its own separate number line, called the **imaginary axis**, which is perpendicular to the usual number line (or **real axis**).

The Mandelbrot set itself is made up of the complex numbers that satisfy a certain rule related to a simple equation. The resulting picture is amazing, and just gets more and more fascinating as you zoom in!



A Mandelbrot set in the Complex plane.

Learning Objectives

Generate fractals given an initiator and generation rule

- Generate a fractal with random variation
- Calculate Fractal Dimension using scaling relation

Complex Numbers

- Identify and make arithmetic calculations with imaginary numbers
- Plot complex numbers on the complex plane
- Define a recursive sequence that will generate a fractal in the complex plane
- Determine whether a complex number is part of the Mandelbrot set

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- Mandelbrot Set. **Authored by:** Lars H. Rohwedder. **Located at:** https://commons.wikimedia.org/wiki/File:Mandelbrot_Set_in_Complex_Plane.png. **License:** Public Domain: No Known Copyright

INTRODUCTION: FRACTAL BASICS

Learning Objectives

By the end of this lesson you will be able to:

- Define and identify self-similarity in geometric shapes, plants, and geological formations
- Generate a fractal shape given an initiator and a generator
- Scale a geometric object by a specific scaling factor using the scaling dimension relation
- Determine the fractal dimension of a fractal object

In the 1980's and 90's fractals were a popular, household idea. Their presence in popular culture may have waned in the last 20 years, but their presence in nature, economics, and nearly everything around us has not. In this section, we will introduce some of the fundamental ideas, and terminology that will help you begin to understand the mathematical principles that define fractals.

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FRACTAL BASICS

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties. We'll explore what that sentence means through the rest of the chapter. For now, we can begin with the idea of self-similarity, a characteristic of most fractals.

Self-similarity

A shape is **self-similar** when it looks essentially the same from a distance as it does closer up.

Self-similarity can often be found in nature. In the Romanesco broccoli pictured below (Note: http://en.wikipedia.org/wiki/File:Cauliflower_Fractal_AVM.JPG), if we zoom in on part of the image, the piece remaining looks similar to the whole.

Likewise, in the fern frond below (Note: <http://www.flickr.com/photos/cjewel/3261398909/>), one piece of the frond looks similar to the whole.

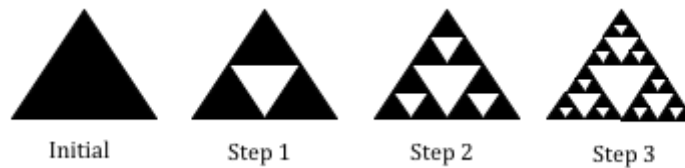
Similarly, if we zoom in on the coastline of Portugal (Note: Openstreetmap.org, CC-BY-SA), each zoom reveals previously hidden detail, and the coastline, while not identical to the view from further way, does exhibit similar characteristics.

Iterated Fractals

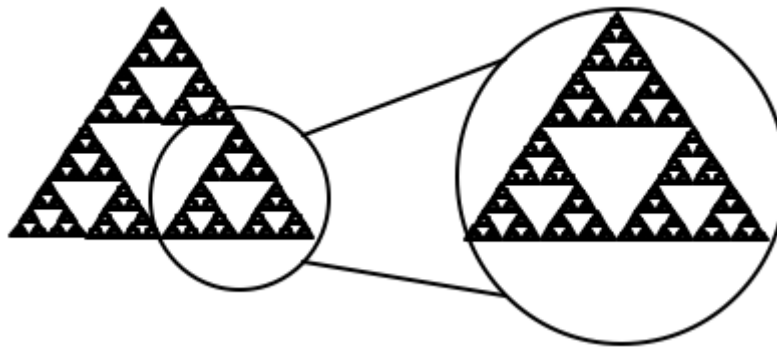
This self-similar behavior can be replicated through recursion: repeating a process over and over.

Example

Suppose that we start with a filled-in triangle. We connect the midpoints of each side and remove the middle triangle. We then repeat this process.



If we repeat this process, the shape that emerges is called the Sierpinski gasket. Notice that it exhibits self-similarity—any piece of the gasket will look identical to the whole. In fact, we can say that the Sierpinski gasket contains three copies of itself, each half as tall and wide as the original. Of course, each of those copies also contains three copies of itself.



In the following video, we present another explanation of how to generate a Sierpinski gasket using the idea of self-similarity.

Watch this video online: <https://youtu.be/vro9BUfJxTA>

We can construct other fractals using a similar approach. To formalize this a bit, we're going to introduce the idea of initiators and generators.

Initiators and Generators

An **initiator** is a starting shape

A **generator** is an arranged collection of scaled copies of the initiator

To generate fractals from initiators and generators, we follow a simple rule:

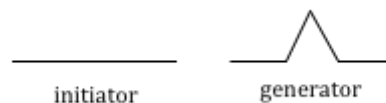
Fractal Generation Rule

At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary

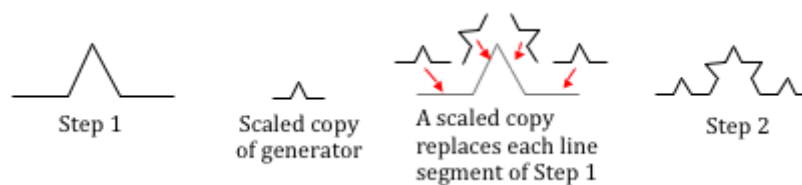
This process is easiest to understand through example.

Example

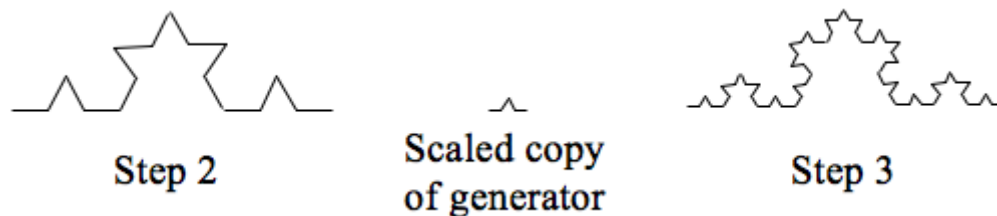
Use the initiator and generator shown to create the iterated fractal.



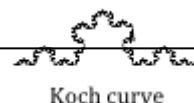
This tells us to, at each step, replace each line segment with the spiked shape shown in the generator. Notice that the generator itself is made up of 4 copies of the initiator. In step 1, the single line segment in the initiator is replaced with the generator. For step 2, each of the four line segments of step 1 is replaced with a scaled copy of the generator:



This process is repeated to form Step 3. Again, each line segment is replaced with a scaled copy of the generator.



Notice that since Step 0 only had 1 line segment, Step 1 only required one copy of Step 0. Since Step 1 had 4 line segments, Step 2 required 4 copies of the generator. Step 2 then had 16 line segments, so Step 3 required 16 copies of the generator. Step 4, then, would require $16 \cdot 4 = 64$ copies of the generator. The shape resulting from iterating this process is called the Koch curve, named for Helge von Koch who first explored it in 1904.



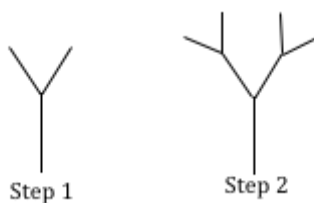
Notice that the Sierpinski gasket can also be described using the initiator-generator approach.

Example

Use the initiator and generator below, however only iterate on the “branches.” Sketch several steps of the iteration.

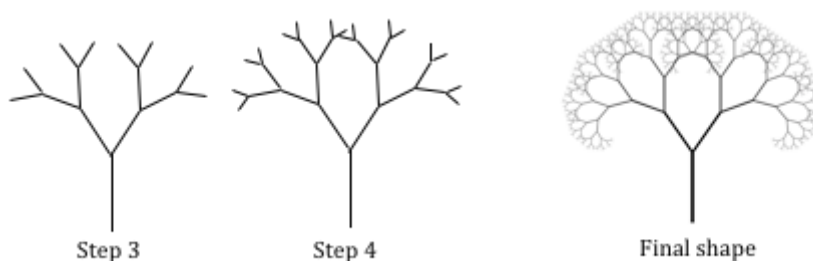


We begin by replacing the initiator with the generator. We then replace each “branch” of Step 1 with a scaled copy of the generator to create Step 2.



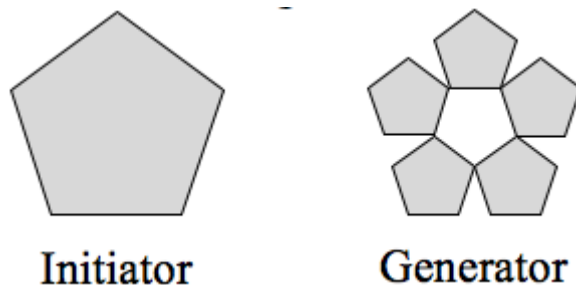
Step 1, the generator. Step 2, one iteration of the generator.

We can repeat this process to create later steps. Repeating this process can create intricate tree shapes. (Note: <http://www.flickr.com/photos/visualarts/5436068969/>)



Try It Now

Use the initiator and generator shown to produce the next two stages.



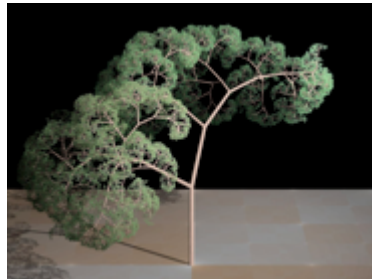
Answer



Step 2



Step 3



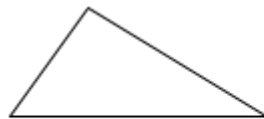
Using iteration processes like those above can create a variety of beautiful images evocative of nature. (Note: http://en.wikipedia.org/wiki/File:Fractal_tree_%28Plate_b_-_2%29.jpg) (Note: http://en.wikipedia.org/wiki/File:Barnsley_Fern_fractals_-_4_states.PNG)

More natural shapes can be created by adding in randomness to the steps.

Example

Create a variation on the Sierpinski gasket by randomly skewing the corner points each time an iteration is made.

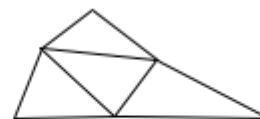
Suppose we start with the triangle below. We begin, as before, by removing the middle triangle. We then add in some randomness.



Step 0

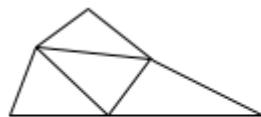


Step 1



Step 1 with
randomness

We then repeat this process.



Step 1 with
randomness

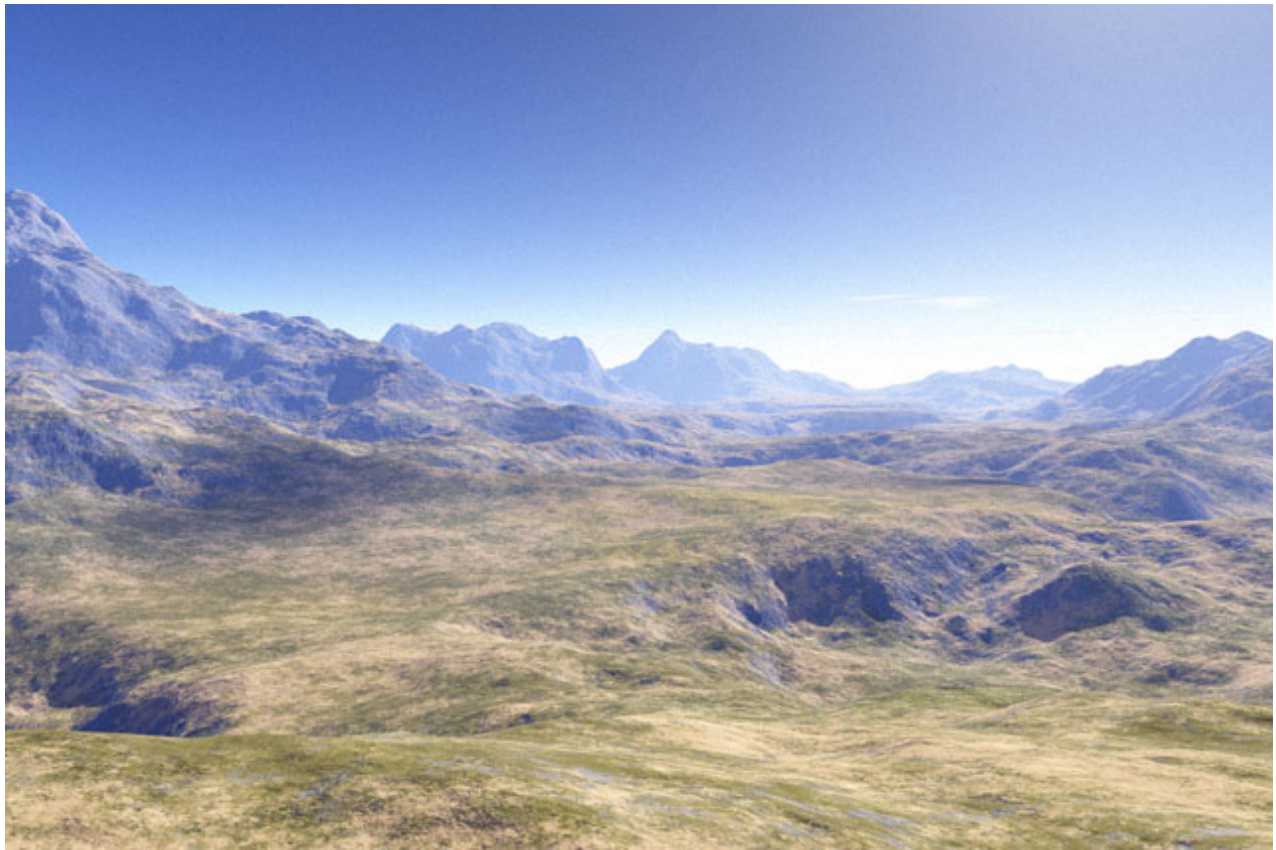


Step 2



Step 2 with
randomness

Continuing this process can create mountain-like structures. This landscape (Note: <http://en.wikipedia.org/wiki/File:FractalLandscape.jpg>) was created using fractals, then colored and textured.



The following video provides another view of branching fractals, and randomness.

Watch this video online: <https://youtu.be/OyAL-66GkJY>

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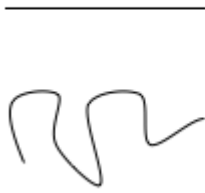
- Self similarity. **Authored by:** OCLPhase2. **Located at:** <https://youtu.be/vro9BUfxTA>. **License:** *CC BY: Attribution*
- Iterated tree and twisted gasket. **Authored by:** OCLPhase2. **Located at:** <https://youtu.be/OyAL-66GkJY>. **License:** *Public Domain: No Known Copyright*

FRACTAL DIMENSION

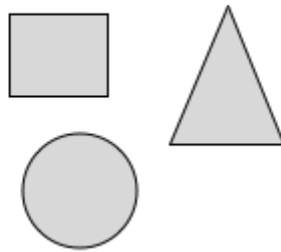
In addition to visual self-similarity, fractals exhibit other interesting properties. For example, notice that each step of the Sierpinski gasket iteration removes one quarter of the remaining area. If this process is continued indefinitely, we would end up essentially removing all the area, meaning we started with a 2-dimensional area, and somehow end up with something less than that, but seemingly more than just a 1-dimensional line.

To explore this idea, we need to discuss dimension. Something like a line is 1-dimensional; it only has length. Any curve is 1-dimensional. Things like boxes and circles are 2-dimensional, since they have length and width, describing an area. Objects like boxes and cylinders have length, width, and height, describing a volume, and are 3-dimensional.

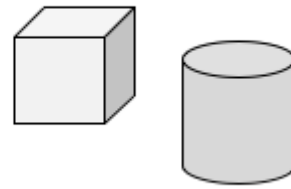
1-dimensional



2-dimensional



3-dimensional

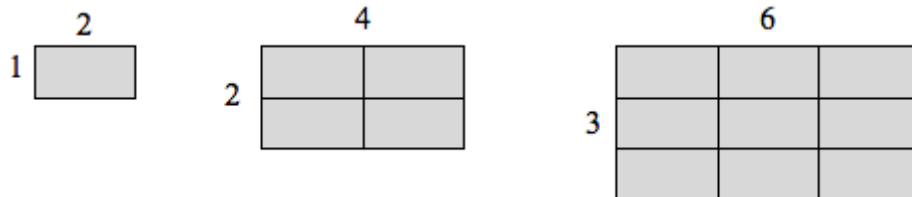


Certain rules apply for scaling objects, related to their dimension.

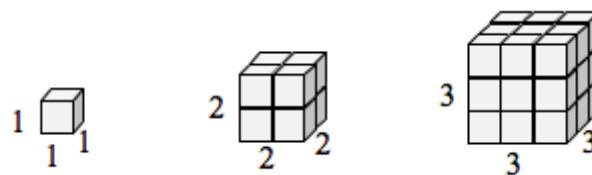
If I had a line with length 1, and wanted to scale its length by 2, I would need two copies of the original line. If I had a line of length 1, and wanted to scale its length by 3, I would need three copies of the original.



If I had a rectangle with length 2 and height 1, and wanted to scale its length and width by 2, I would need four copies of the original rectangle. If I wanted to scale the length and width by 3, I would need nine copies of the original rectangle.



If I had a cubical box with sides of length 1, and wanted to scale its length and width by 2, I would need eight copies of the original cube. If I wanted to scale the length and width by 3, I would need 27 copies of the original cube.



Notice that in the 1-dimensional case, copies needed = scale.

In the 2-dimensional case, copies needed = scale^2 .

In the 3-dimensional case, copies needed = scale^3 .

From these examples, we might infer a pattern.

Scaling-Dimension Relation

To scale a D -dimensional shape by a scaling factor S , the number of copies C of the original shape needed will be given by:

$$\text{Copies} = \text{Scale}^{\text{Dimension}}, \text{ or } C = S^D$$

Example

Use the scaling-dimension relation to determine the dimension of the Sierpinski gasket.

Suppose we define the original gasket to have side length 1. The larger gasket shown is twice as wide and twice as tall, so has been scaled by a factor of 2.



Notice that to construct the larger gasket, 3 copies of the original gasket are needed.

Using the scaling-dimension relation $C = S^D$, we obtain the equation $3 = 2^D$.

Since $2^1 = 2$ and $2^2 = 4$, we can immediately see that D is somewhere between 1 and 2; the gasket is more than a 1-dimensional shape, but we've taken away so much area its now less than 2-dimensional.

Solving the equation $3 = 2^D$ requires logarithms. If you studied logarithms earlier, you may recall how to solve this equation (if not, just skip to the box below and use that formula with the log key on a calculator): Take the logarithm of both sides.

$$3 = 2^D$$

Use the exponent property of logs.

$$\log(3) = \log(2^D)$$

Divide by $\log(2)$.

$$\log(3) = D \log(2)$$

The dimension of the gasket is about 1.585.

$$D = \frac{\log(3)}{\log(2)} \approx 1.585$$

Scaling-Dimension Relation, to find Dimension

To find the dimension D of a fractal, determine the scaling factor S and the number of copies C of the original shape needed, then use the formula

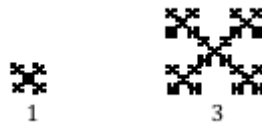
$$D = \frac{\log(C)}{\log(S)}$$

Try It Now

Determine the fractal dimension of the fractal produced using the initiator and generator.



Answer



Scaling the fractal by a factor of 3 requires 5 copies of the original.

$$D = \frac{\log(5)}{\log(3)} \approx 1.465$$

Visit this page in your course online to practice before taking the quiz.

In the following video, we present a worked example of how to determine the dimension of the Sierpinski gasket

Watch this video online: <https://youtu.be/B1WTSsuDvWc>

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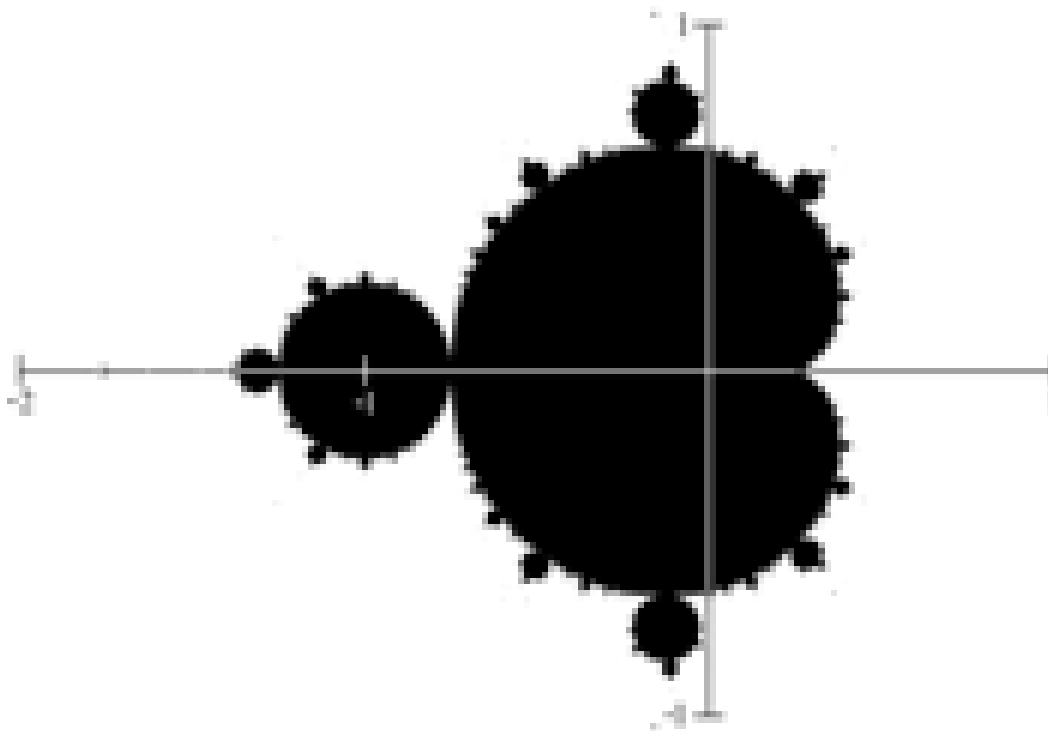
INTRODUCTION: FRACTALS GENERATED BY COMPLEX NUMBERS

Learning Objectives

The learning objectives for this lesson include:

- Identify the difference between an imaginary number and a complex number
- Identify the real and imaginary parts of a complex number
- Plot a complex number on the complex plane
- Perform arithmetic operations on complex numbers
- Graph physical representations of arithmetic operations on complex numbers as scaling or rotation
- Generate several terms of a recursive relation
- Determine whether a complex number is part of the set of numbers that make up the Mandelbrot set

You may be familiar with the fractal in the image below. The boundary of this shape exhibits quasi-self-similarity, in that portions look very similar to the whole. This object is called the Mandelbrot set and is generated by iterating a simple recursive rule using complex numbers. In this lesson, you will first learn about the arithmetic of complex numbers so you can understand how a fractal like the Mandelbrot set is generated.



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ARITHMETIC WITH COMPLEX NUMBERS

Complex Numbers

(Note: Portions of this section are remixed from Precalculus: An Investigation of Functions by David Lippman and Melonie Rasmussen. CC-BY-SA)

The numbers you are most familiar with are called **real numbers**. These include numbers like 4, 275, -200, 10.7, $\frac{1}{2}$, π , and so forth. All these real numbers can be plotted on a number line. For example, if we wanted to show the number 3, we plot a point:

To solve certain problems like $x^2 = -4$, it became necessary to introduce **imaginary numbers**.

Imaginary Number i

The imaginary number i is defined to be $i = \sqrt{-1}$.
Any real multiple of i , like $5i$, is also an imaginary number.

Example

Simplify $\sqrt{-9}$.

We can separate $\sqrt{-9}$ as $\sqrt{9}\sqrt{-1}$. We can take the square root of 9, and write the square root of -1 as i .
 $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$

A complex number is the sum of a real number and an imaginary number.

Complex Number

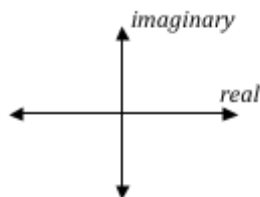
A **complex number** is a number $z = a + bi$, where

- a and b are real numbers
- a is the real part of the complex number
- b is the imaginary part of the complex number

To plot a complex number like $3 - 4i$, we need more than just a number line since there are two components to the number. To plot this number, we need two number lines, crossed to form a complex plane.

Complex Plane

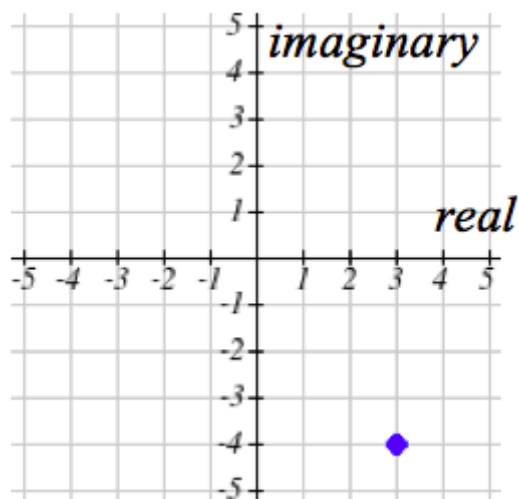
In the **complex plane**, the horizontal axis is the real axis and the vertical axis is the imaginary axis.



Example

Plot the number $3 - 4i$ on the complex plane.

The real part of this number is 3, and the imaginary part is -4 . To plot this, we draw a point 3 units to the right of the origin in the horizontal direction and 4 units down in the vertical direction.



Try It Now

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Because this is analogous to the Cartesian coordinate system for plotting points, we can think about plotting our complex number $z = a + bi$ as if we were plotting the point (a, b) in Cartesian coordinates. Sometimes people write complex numbers as $z = x + yi$ to highlight this relation.

Arithmetic on Complex Numbers

Before we dive into the more complicated uses of complex numbers, let's make sure we remember the basic arithmetic involved. To add or subtract complex numbers, we simply add the like terms, combining the real parts and combining the imaginary parts.

Example

Add $3 - 4i$ and $2 + 5i$.

Adding $(3 - 4i) + (2 + 5i)$, we add the real parts and the imaginary parts.

$$3 + 2 - 4i + 5i$$

$$5 + i$$

Try It Now

Subtract $2 + 5i$ from $3 - 4i$.

Answer

$$(3 - 4i) - (2 + 5i) = 1 - 9i$$

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In the following video, we present more worked examples of arithmetic with complex numbers.

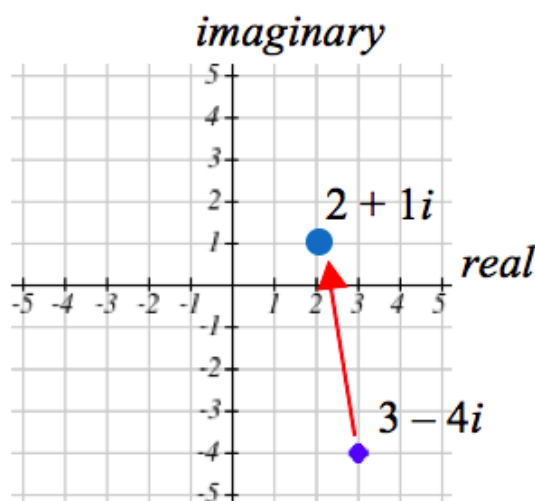
Watch this video online: <https://youtu.be/XJXDcybM84U>

When we add complex numbers, we can visualize the addition as a shift, or translation, of a point in the complex plane.

Example

Visualize the addition $3 - 4i$ and $-1 + 5i$.

The initial point is $3 - 4i$. When we add $-1 + 5i$, we add -1 to the real part, moving the point 1 units to the left, and we add 5 to the imaginary part, moving the point 5 units vertically. This shifts the point $3 - 4i$ to $2 + 1i$.



Try It Now

Visit this page in your course online to practice before taking the quiz.

We can also multiply complex numbers by a real number, or multiply two complex numbers.

Example

Multiply: $4(2 + 5i)$

To multiply the complex number by a real number, we simply distribute as we would when multiplying polynomials.

Distribute and simplify.

$$4(2 + 5i)$$

$$= 4 \cdot 2 + 4 \cdot 5i$$

$$= 8 + 20i$$

Example

Multiply: $(2 + 5i)(4 + i)$.

$$(2 + 5i)(4 + i)$$

$$= 8 + 20i + 2i + 5i^2$$

$$= 8 + 20i + 2i + 5(-1)$$

$$= 3 + 22i$$

Expand.

Since $i = \sqrt{-1}$, $i^2 = -1$

Simplify.

Try It Now

Multiply $3 - 4i$ and $2 + 3i$.

Answer

Multiply $(3 - 4i)(2 + 3i) = 6 + 9i - 8i - 12i^2 = 6 + i - 12(-1) = 18 + i$

Visit this page in your course online to practice before taking the quiz.

To understand the effect of multiplication visually, we'll explore three examples.

Example

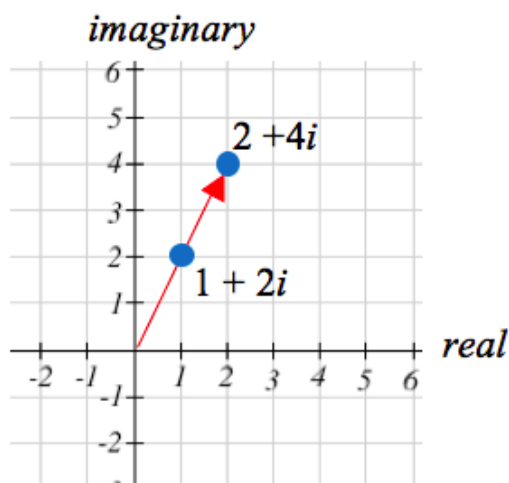
Visualize the product $2(1 + 2i)$.

Multiplying we'd get

$$2 \cdot 1 + 2 \cdot 2i$$

$$= 2 + 4i$$

Notice both the real and imaginary parts have been scaled by 2. Visually, this will stretch the point outwards, away from the origin.



Example

Visualize the product $i(1 + 2i)$.

Multiplying, we'd get

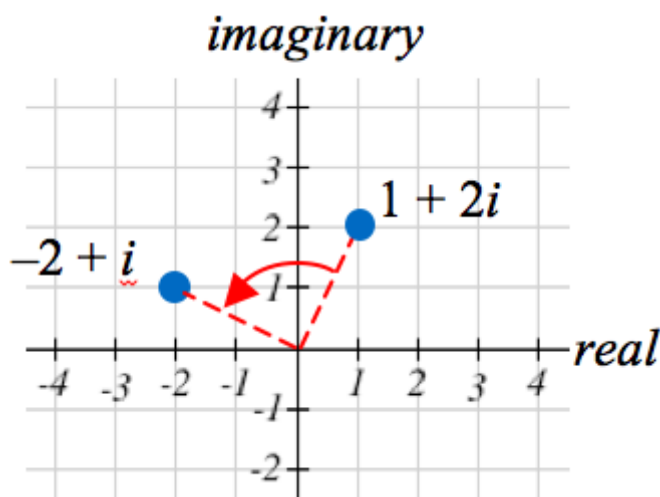
$$i \cdot 1 + i \cdot 2i$$

$$= i + 2i^2$$

$$= i + 2(-1)$$

$$= -2 + i$$

In this case, the distance from the origin has not changed, but the point has been rotated about the origin, 90° counter-clockwise.



Try It Now

Multiply $3 - 4i$ and $2 + 3i$.

Answer

$$\text{Multiply } (3 - 4i)(2 + 3i) = 6 + 9i - 8i - 12i^2 = 6 + i - 12(-1) = 18 + i$$

Visit this page in your course online to practice before taking the quiz.

Example

Visualize the result of multiplying $1 + 2i$ by $1 + i$. Then show the result of multiplying by $1 + i$ again.

Multiplying $1 + 2i$ by $1 + i$,

$$-4 + 2i$$

$$(1 + 2i)(1 + i)$$

$$= 1 + i + 2i + 2i^2$$

$$= 1 + 3i + 2(-1)$$

$$= -1 + 3i$$

Multiplying by $1 + i$ again,

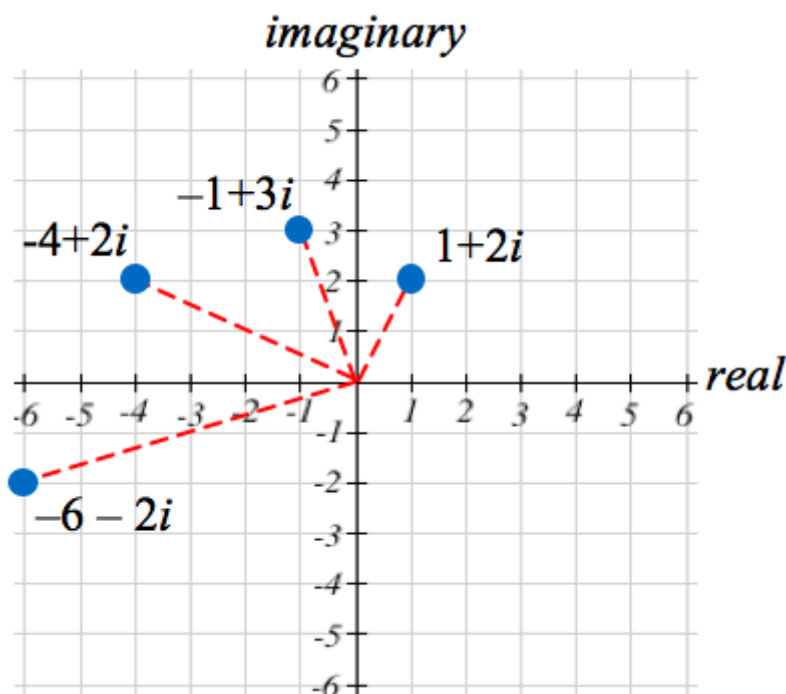
$$(-1 + 3i)(1 + i)$$

$$= -1 - i + 3i + 3i^2$$

$$= -1 + 2i + 3(-1)$$

$$= -4 + 2i$$

If we multiplied by $1 + i$ again, we'd get $-6 - 2i$. Plotting these numbers in the complex plane, you may notice that each point gets both further from the origin, and rotates counterclockwise, in this case by 45° .



In general, multiplication by a complex number can be thought of as a **scaling**, changing the distance from the origin, combined with a **rotation** about the origin.

Try It Now

Visit this page in your course online to practice before taking the quiz.

The following video presents more examples of how to visualize the results of arithmetic on complex numbers.

In the following video, we present more worked examples of arithmetic with complex numbers.

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GENERATING FRACTALS WITH COMPLEX NUMBERS

Complex Recursive Sequences

Some fractals are generated with complex numbers. The Mandelbrot set, which we introduced briefly at the beginning of this module, is generated using complex numbers with a recursive sequence. Before we can see how to generate the Mandelbrot set, we need to understand what a recursive sequence is.

Recursive Sequence

A **recursive relationship** is a formula which relates the next value, z_{n+1} , in a sequence to the previous value, z_n . In addition to the formula, we need an initial value, z_0 .

The sequence of values produced is the recursive sequence.

Example

Given the recursive relationship $z_{n+1} = z_n + 2$, $z_0 = 4$, generate several terms of the recursive sequence.

We are given the starting value, $z_0 = 4$. The recursive formula holds for any value of n , so if $n = 0$, then $z_{n+1} = z_n + 2$ would tell us $z_{0+1} = z_0 + 2$, or more simply, $z_1 = z_0 + 2$.

Notice this defines z_1 in terms of the known z_0 , so we can compute the value:

$$z_1 = z_0 + 2 = 4 + 2 = 6.$$

Now letting $n = 1$, the formula tells us $z_{1+1} = z_1 + 2$, or $z_2 = z_1 + 2$. Again, the formula gives the next value in the sequence in terms of the previous value.

$$z_2 = z_1 + 2 = 6 + 2 = 8$$

Continuing,

$$z_3 = z_2 + 2 = 8 + 2 = 10$$

$$z_4 = z_3 + 2 = 10 + 2 = 12$$

Try It Now

Visit this page in your course online to practice before taking the quiz.

The previous example generated a basic linear sequence of real numbers. The same process can be used with complex numbers.

Example

Given the recursive relationship $z_{n+1} = z_n \cdot i + (1 - i)$, $z_0 = 4$, generate several terms of the recursive sequence.

We are given $z_0 = 4$. Using the recursive formula:

$$z_1 = z_0 \cdot i + (1 - i) = 4 \cdot i + (1 - i) = 1 + 3i$$

$$z_2 = z_1 \cdot i + (1 - i) = (1 + 3i) \cdot i + (1 - i) = i + 3i^2 + (1 - i) = i - 3 + (1 - i) = -2$$

$$z_3 = z_2 \cdot i + (1 - i) = (-2) \cdot i + (1 - i) = -2i + (1 - i) = 1 - 3i$$

$$z_4 = z_3 \cdot i + (1 - i) = (1 - 3i) \cdot i + (1 - i) = i - 3i^2 + (1 - i) = i + 3 + (1 - i) = 4$$

$$z_5 = z_4 \cdot i + (1 - i) = 4 \cdot i + (1 - i) = 1 + 3i$$

Notice this sequence is exhibiting an interesting pattern—it began to repeat itself.

Try It Now

Visit this page in your course online to practice before taking the quiz.

In the following video we show more worked examples of how to generate the terms of a recursive, complex sequence.

Watch this video online: <https://youtu.be/OyusyTslTs>

Mandelbrot Set

The Mandelbrot Set is a set of numbers defined based on recursive sequences.

Mandelbrot Set

For any complex number c , define the sequence $z_{n+1} = z_n^2 + c$, $z_0 = 0$

If this sequence always stays close to the origin (within 2 units), then the number c is part of the **Mandelbrot Set**. If the sequence gets far from the origin, then the number c is not part of the set.

Example

Determine if $c = 1 + i$ is part of the Mandelbrot set.

We start with $z_0 = 0$. We continue, omitting some detail of the calculations

$$z_1 = z_0^2 + 1 + i = 0 + 1 + i = 1 + i$$

$$z_2 = z_1^2 + 1 + i = (1 + i)^2 + 1 + i = 1 + 3i$$

$$z_3 = z_2^2 + 1 + i = (1 + 3i)^2 + 1 + i = -7 + 7i$$

$$z_4 = z_3^2 + 1 + i = (-7 + 7i)^2 + 1 + i = 1 - 97i$$

We can already see that these values are getting quite large. It does not appear that $c = 1 + i$ is part of the Mandelbrot set.

Example

Determine if $c = 0.5i$ is part of the Mandelbrot set.

We start with $z_0 = 0$. We continue, omitting some detail of the calculations

$$z_1 = z_0^2 + 0.5i = 0 + 0.5i = 0.5i$$

$$z_2 = z_1^2 + 0.5i = (0.5i)^2 + 0.5i = -0.25 + 0.5i$$

$$z_3 = z_2^2 + 0.5i = (-0.25 + 0.5i)^2 + 0.5i = -0.1875 + 0.25i$$

$$z_4 = z_3^2 + 0.5i = (-0.1875 + 0.25i)^2 + 0.5i = -0.02734 + 0.40625i$$

While not definitive with this few iterations, it does appear that this value is remaining small, suggesting that $0.5i$ is part of the Mandelbrot set.

Try It Now

Determine if $c = 0.4 + 0.3i$ is part of the Mandelbrot set.

Answer

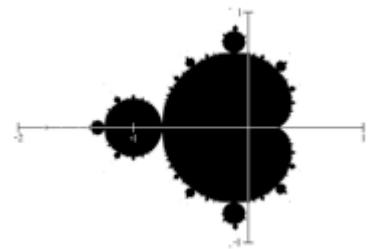
$$z_1 = z_0^2 + 0.4 + 0.3i = 0 + 0.4 + 0.3i = 0.4 + 0.3i$$

$$z_2 = z_1^2 + 0.4 + 0.3i = (0.4 + 0.3i)^2 + 0.4 + 0.3i$$

$$z_3 = z_2^2 + 0.5i = (-0.25 + 0.5i)^2 + 0.5i = -0.1875 + 0.25i$$

$$z_4 = z_3^2 + 0.5i = (-0.1875 + 0.25i)^2 + 0.5i = -0.02734 + 0.40625i$$

If all complex numbers are tested, and we plot each number that is in the Mandelbrot set on the complex plane, we obtain the shape to the right.
(Note: http://en.wikipedia.org/wiki/File:Mandelset_hires.png)



The boundary of this shape exhibits quasi-self-similarity, in that portions look very similar to the whole.

Watch the following video for more examples of how to determine whether a complex number is a member of the Mandelbrot set.

Watch this video online: <https://youtu.be/ORqk5jAFpWg>

In addition to coloring the Mandelbrot set itself black, it is common to color the points in the complex plane surrounding the set. To create a meaningful coloring, often people count the number of iterations of the recursive sequence that are required for a point to get further than 2 units away from the origin. For example, using $c = 1 + i$ above, the sequence was distance 2 from the origin after only two recursions.

For some other numbers, it may take tens or hundreds of iterations for the sequence to get far from the origin. Numbers that get big fast are colored one shade, while colors that are slow to grow are colored another shade. For example, in the image below (Note: This series was generated using Scott's Mandelbrot Set Explorer), light blue is used for numbers that get large quickly, while darker shades are used for numbers that grow more

slowly. Greens, reds, and purples can be seen when we zoom in—those are used for numbers that grow very slowly.

The Mandelbrot set, for having such a simple definition, exhibits immense complexity. Zooming in on other portions of the set yields fascinating swirling shapes.

Additional Resources

A much more extensive coverage of fractals can be found on the [Fractal Geometry site](#). This site includes links to several Java software programs for exploring fractals.

If you are impressed with the Mandelbrot set, [check out this TED talk from 2010](#) given by Benoit Mandelbrot on fractals and the art of roughness.

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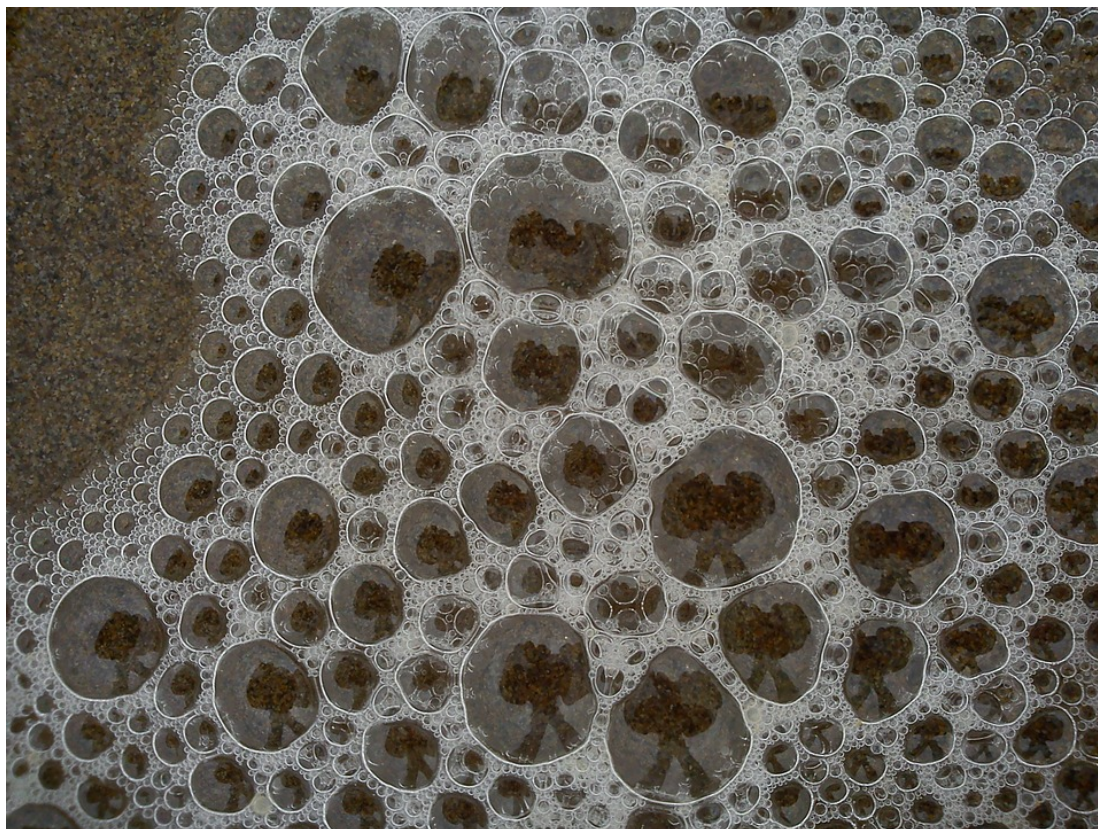
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PUTTING IT TOGETHER: FRACTALS



Foam consists of bubbles packed together in a fractal pattern.

Let's use what we have learned about fractals to study a real-world phenomenon: foam. By definition a **foam** is any material made up of bubbles packed closely together. If the bubbles tend to be very large, you might call it a **froth**.

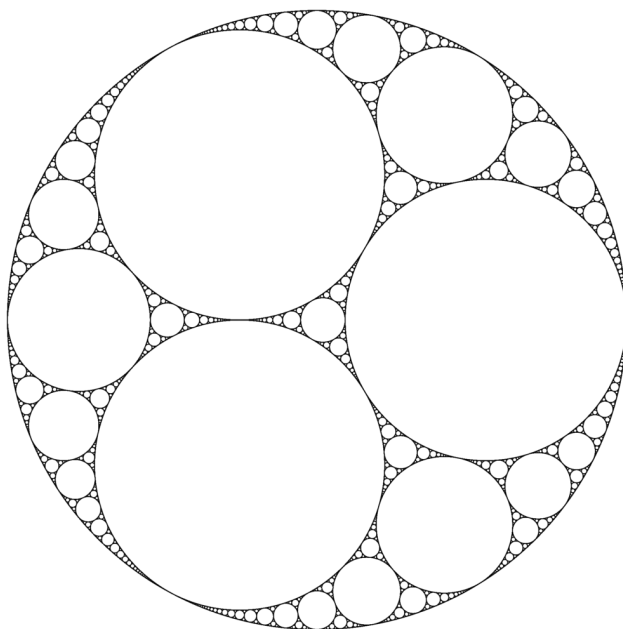
We know that bubbles like to form spheres, because the sphere is the most efficient shape for minimizing surface area around a fixed volume. When you blow a soap bubble on a warm spring day, for example, that bubble will be approximately spherical until it pops.

However, spheres do not pack together very nicely. There's always gaps between the adjacent spheres. So when a foam forms, there may be some number of large bubbles, interspersed with smaller bubbles in the gaps, which in turn have even smaller bubbles in their gaps and so on. The foam is approximately self-similar on smaller and smaller scales; in other words, foam is fractal.

Let's take a look at a two-dimensional idealized version of foam called the **Apollonian gasket**. This figure is created by the following procedure. It helps to have a compass handy.

1. Draw a large circle.
2. Within the circle, draw three smaller circles that all touch one another. In technical terms, we say that the circles are **mutually tangent** to one another.
3. In the gaps between these circles, draw smaller circles that are as large as possible without overlapping any existing circles. If you do this correctly, the new circle will be tangent to two of the circles from step 2 as well as the original big circle.
4. Continue in this way, filling each new gap with as large a circle that will fit without overlap. Notice that with each new circle, there will be multiple new gaps to fill.

The process should continue indefinitely, however you will eventually reach a stage in which the gaps are smaller than the width of your pencil or pen. At that point, you can step back and admire your work. A computer-generated Apollonian gasket is shown in the figure below.



The Apollonian gasket is a fractal that can be used to model soap bubble foam.

Because the Apollonian gasket is only approximately self-similar, there is not a well-defined scaling-dimension. However, if you look at any “triangular” section within three circles, it looks like a curved version of the Sierpinski gasket. Recall, it requires 3 copies of the Sierpinski gasket in order to scale it by a factor of 2. So we would expect the fractal dimension of the Apollonian gasket to be close to:

$$D = \frac{\log(3)}{\log(2)} \approx 1.585$$

In fact, using a more general definition of fractal dimension, it can be shown that the dimension of the Apollonian gasket is about 1.3057. This implies that the gasket is somehow closer to being one-dimensional than two-dimensional. In turn, a foam made of large bubbles, like the froth on top of your latte is more two-dimensional than three-dimensional. Remember this next time you get an extremely frothy drink; there’s very little substance to it!

If you would like a more detailed instructions on how to make your own Apollonian gasket, they are provided at the following website: <http://www.wikihow.com/Create-an-Apollonian-Gasket>

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SET THEORY AND LOGIC

WHY IT MATTERS: SET THEORY AND LOGIC

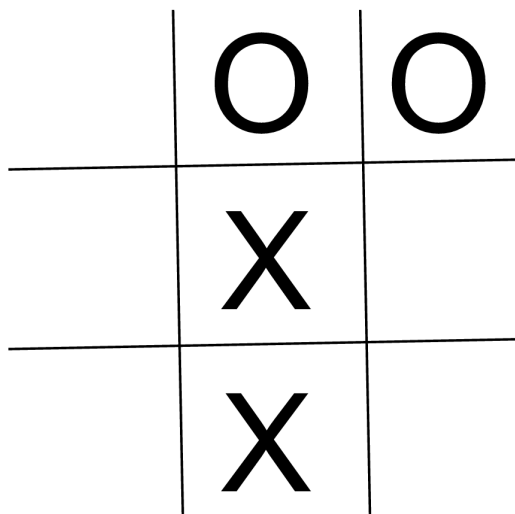
Let’s play a game!

Almost everyone knows the game of Tic-Tac-Toe, in which players mark X’s and O’s on a three-by-three grid until one player makes three in a row, or the grid gets filled up with no winner (a draw). The rules are so simple that kids as young as 3 or 4 can get the idea.

At first, a young child may play haphazardly, marking the grid without thinking about how the other player might respond. For example, the child might eagerly make two in a row but fail to see that his older sister will be able to complete three in a row on her next turn.

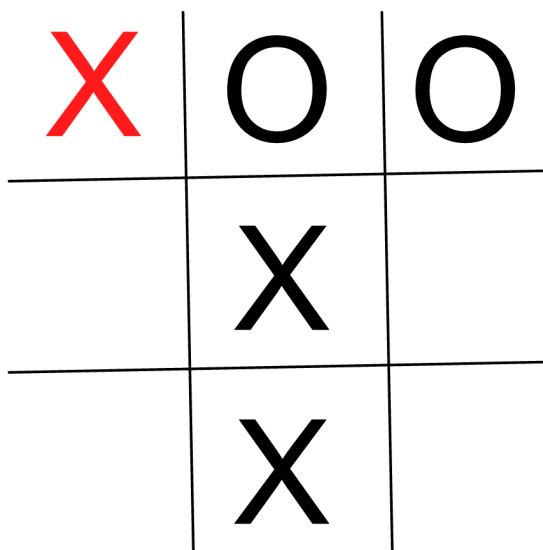
It’s not until about age 6 or so that children begin to strategize, looking at their opponent’s potential moves and responses. The child begins to use systematic reasoning, or what we call logic, to decide what will happen in the game if one move is chosen over another.

The logic involved can be fairly complex, especially for a young child. For example, suppose it's your turn (X's), and the grid currently looks like this. Where should you play?



Your thought process (or what we call a logical argument) might go something like this:

- It takes three in a row to win the game.
- I cannot make three in a row no matter where I play on this turn.
- If it were my opponent's turn, then she could make three in a row by putting an O in the upper left corner.
- If I don't put my X in the upper left corner, then my opponent will have the opportunity to play there.
- Therefore, I must put an X in the upper left corner.



Because you are much more experienced than the typical 6 year-old child, I bet that you immediately saw where the X should be played, even without thinking through all of the details listed above. In fact, if you have played a fair number of Tic-Tac-Toe games in your childhood, then there are neural pathways in your brain that are hard-wired for Tic-Tac-Toe logic, just like a computer might be hard-wired to complete certain routine tasks.

Indeed, computers follow the rules of logic by design. Certain components called gates shunt electricity in various ways throughout the circuitry of the computer, allowing it to perform whatever procedures it is programmed to do.

So, whether you are trying to find the winning Tic-Tac-Toe strategy, putting together a valid argument to convince fellow lawmakers to preserve important funding, or designing powerful computers to help solve complicated problems, logic is an essential part of our world.

Learning Objectives

Organize Sets and Use Sets to Describe Relationships

- Describe memberships of sets, including the empty set, using proper notation, and decide whether given items are members and determine the cardinality of a given set
- Perform the operations of union, intersection, complement, and difference on sets using proper notation
- Describe the relations between sets regarding membership, equality, subset, and proper subset, using proper notation
- Be able to draw and interpret Venn diagrams of set relations and operations and use Venn diagrams to solve problems
- Recognize when set theory is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others

Introduction to Logic

- Combine sets using Boolean logic, using proper notations
- Use statements and conditionals to write and interpret expressions
- Use a truth table to interpret complex statements or conditionals
- Write truth tables given a logical implication, and it's related statements – converse, inverse, and contrapositive
- Determine whether two statements are logically equivalent
- Use DeMorgan's laws to define logical equivalences of a statement

Analyzing Arguments With Logic

- Discern between an inductive argument and a deductive argument
- Evaluate deductive arguments
- Analyze arguments with Venn diagrams and truth tables
- Use logical inference to infer whether a statement is true
- Identify logical fallacies in common language including appeal to ignorance, appeal to authority, appeal to consequence, false dilemma, circular reasoning, post hoc, correlation implies causation, and straw man arguments

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INTRODUCTION: SET THEORY BASICS

Learning Objectives

- Describe memberships of sets, including the empty set, using proper notation, and decide whether given items are members and determine the cardinality of a given set.
- Describe the relations between sets regarding membership, equality, subset, and proper subset, using proper notation.
- Perform the operations of union, intersection, complement, and difference on sets using proper notation.
- Be able to draw and interpret Venn diagrams of set relations and operations and use Venn diagrams to solve problems.
- Recognize when set theory is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

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INTRODUCTION TO SET THEORY

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze data.

Introduction

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set
A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example

Some examples of sets defined by describing the contents:

1. The set of all even numbers
2. The set of all books written about travel to Chile

Answer

Some examples of sets defined by listing the elements of the set:

1. $\{1, 3, 9, 12\}$
2. $\{\text{red, orange, yellow, green, blue, indigo, purple}\}$

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol \in means “is an element of”.

A set that contains no elements, $\{ \}$, is called the **empty set** and is notated \emptyset

Example

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Key Takeaways

Visit this page in your course online to practice before taking the quiz.

A set simply specifies the contents; order is not important. The set represented by $\{1, 2, 3\}$ is equivalent to the set $\{3, 1, 2\}$.

Subsets

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A , but may not contain all the elements of A .

If B is a subset of A , we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.

If B is a proper subset of A , we write $B \subset A$

Example

Consider these three sets:

A = the set of all even numbers

$B = \{2, 4, 6\}$

$C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A .

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A , since C contains an element, 3, that is not contained in A

Watch this video online: https://youtu.be/5xthPHH4i_A

Example

Suppose a set contains the plays “Much Ado About Nothing,” “MacBeth,” and “A Midsummer’s Night Dream.” What is a larger set this might be a subset of?

Answer

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try It Now

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Answer

Put Answer Here

Exercises

Given the set: $A = \{a, b, c, d\}$. List all of the subsets of A

Answer

$\{\}$ (or \emptyset), $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{a,b,c,d\}$
You can see that there are 16 subsets, 15 of which are proper subsets.

Listing the sets is fine if you have only a few elements. However, if we were to list all of the subsets of a set containing many elements, it would be quite tedious. Instead, in the next example we will consider each element of the set separately.

Example

In the previous example, there are four elements. For the first element, a , either it's in the set or it's not. Thus there are 2 choices for that first element. Similarly, there are two choices for b —either it's in the set or it's not. Using just those two elements, list all the possible subsets of the set $\{a,b\}$

Answer

$\{\}$ —both elements are not in the set

$\{a\}$ — a is in; b is not in the set

$\{b\}$ — a is not in the set; b is in

$\{a,b\}$ — a is in; b is in

Two choices for a times the two for b gives us $2^2 = 4$ subsets.

Now let's include c , just for fun. List all the possible subsets of the new set $\{a,b,c\}$.

Again, either c is included or it isn't, which gives us two choices. The outcomes are $\{\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$. Note that there are $2^3 = 8$ subsets.

If you include four elements, there would be $2^4 = 16$ subsets. 15 of those subsets are proper, 1 subset, namely $\{a,b,c,d\}$, is not.

In general, if you have n elements in your set, then there are 2^n subsets and $2^n - 1$ proper subsets.

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UNION, INTERSECTION, AND COMPLEMENT

Commonly, sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$. More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$. More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.

The **complement** of a set A contains everything that is *not* in the set A . The complement is notated A' , or A^c , or sometimes $\sim A$.

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A .

Example

1. If we were discussing searching for books, the universal set might be all the books in the library.
2. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
3. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then $A^c = \{3, 5, 6, 7, 8, 9\}$.

Try It Now

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Example

Consider the sets:

$A = \{\text{red, green, blue}\}$

$B = \{\text{red, yellow, orange}\}$

$C = \{\text{red, orange, yellow, green, blue, purple}\}$

Find the following:

1. Find $A \cup B$
2. Find $A \cap B$

3. Find $A^c \cap C$

Answer

1. The union contains all the elements in either set: $A \cup B = \{\text{red, green, blue, yellow, orange}\}$ Notice we only list red once.
2. The intersection contains all the elements in both sets: $A \cap B = \{\text{red}\}$
3. Here we're looking for all the elements that are *not* in set A and are also in C . $A^c \cap C = \{\text{orange, yellow, purple}\}$

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Notice that in the example above, it would be hard to just ask for A^c , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example

Suppose $H = \{\text{cat, dog, rabbit, mouse}\}$, $F = \{\text{dog, cow, duck, pig, rabbit}\}$, and $W = \{\text{duck, rabbit, deer, frog, mouse}\}$

1. Find $(H \cap F) \cup W$
2. Find $H \cap (F \cup W)$
3. Find $(H \cap F)^c \cap W$

Answer

1. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$. Now we union that result with W : $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse}\}$
2. We start with the union: $F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse}\}$. Now we intersect that result with H : $H \cap (F \cup W) = \{\text{dog, rabbit, mouse}\}$
3. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$. Now we want to find the elements of W that are *not* in $H \cap F$. $(H \cap F)^c \cap W = \{\text{duck, deer, frog, mouse}\}$

Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

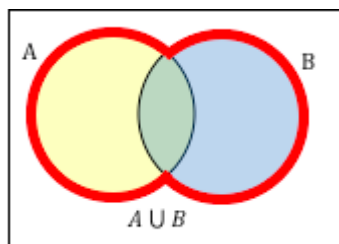
Venn Diagram

A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets. Basic Venn diagrams can illustrate the interaction of two or three sets.

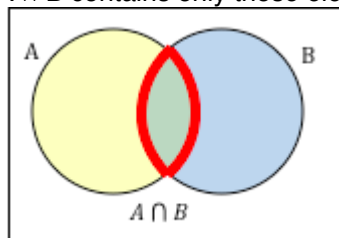
Example

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$
 $A \cup B$ contains all elements in *either* set.

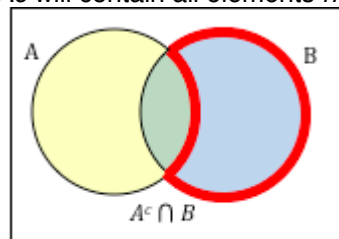
Answer



$A \cap B$ contains only those elements in both sets – in the overlap of the circles.



A^c will contain all elements *not* in the set A . $A^c \cap B$ will contain the elements in set B that are not in set A .

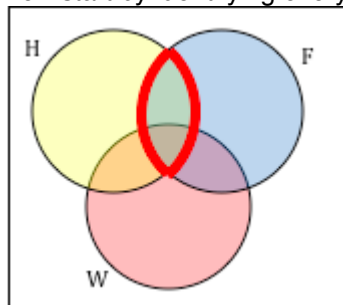


Example

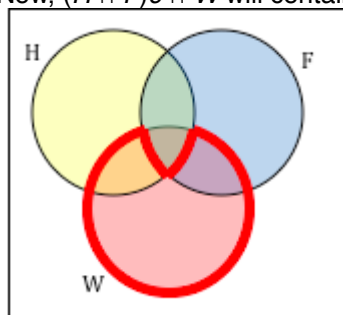
Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

Answer

We'll start by identifying everything in the set $H \cap F$



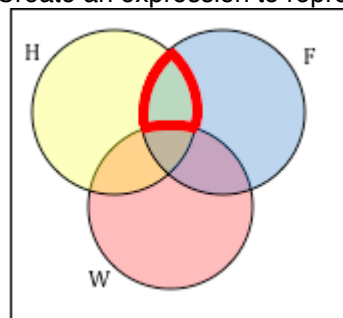
Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W .



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Example

Create an expression to represent the outlined part of the Venn diagram shown.

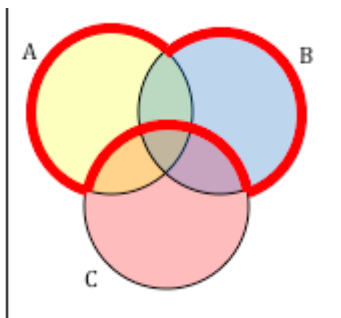


Answer

The elements in the outlined set *are* in sets H and F , but are not in set W . So we could represent this set as $H \cap F \cap W^c$

Try It Now

Create an expression to represent the outlined portion of the Venn diagram shown.



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CARDINALITY

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.
The cardinality of the set A is often notated as $|A|$ or $n(A)$

Exercises

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.
What is the cardinality of B ? $A \cup B$, $A \cap B$?

Answer

The cardinality of B is 4, since there are 4 elements in the set.
The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.
The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

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Exercises

What is the cardinality of P = the set of English names for the months of the year?

Answer

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Exercises

A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

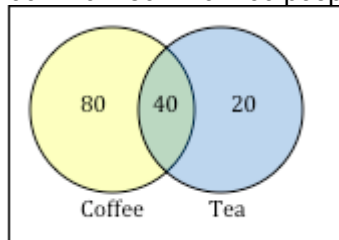
Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Answer

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$200 - 20 - 80 - 40 = 60$ people who drink neither.



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Example

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Answer

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T , then subtract those in intersection that we've counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Example

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$.

Answer

Since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)^c$ must be the other 10%.

The previous example illustrated two important properties called cardinality properties:

Cardinality properties

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A^c) = n(U) - n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Example

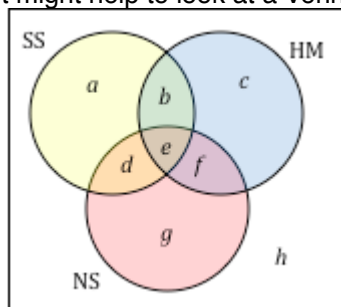
Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Answer

It might help to look at a Venn diagram.



From the given data, we know that there are
3 students in region e and
7 students in region h .

Since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. Since we know there are 3 students in region 3, there must be
 $7 - 3 = 4$ students in region d .

Similarly, since there are 10 students taking HM and NS, which includes regions e and f , there must be
 $10 - 3 = 7$ students in region f .

Since 9 students were taking SS and HM, there must be $9 - 3 = 6$ students in region b .

Now, we know that 21 students were taking a SS course. This includes students from regions a , b , d , and e . Since we know the number of students in all but region a , we can determine that $21 - 6 - 4 - 3 = 8$ students are in region a .

8 students are taking only a SS course.

Try It Now

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

Answer

1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

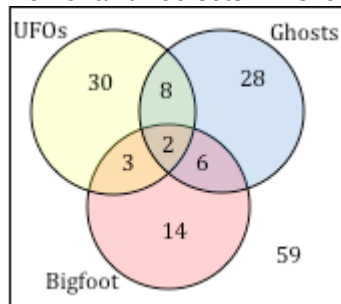
2. $A \cup C = \{\text{red, orange, yellow, green, blue purple}\}$

$B \cap A = \{\text{green, blue}\}$

3. $A \cup B \cap C$

4. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our

way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves $150 - 91 = 59$ who believe in none.



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Watch this video online: <https://youtu.be/wErcETeKvrU>

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INTRODUCTION: LOGIC BASICS

Learning Objectives

Introduction to Logic

- Combine sets using Boolean logic, using proper notations
- Use statements and conditionals to write and interpret expressions
- Use a truth table to interpret complex statements or conditionals
- Write truth tables given a logical implication, and it's related statements – converse, inverse, and contrapositive
- Determine whether two statements are logically equivalent
- Use DeMorgan's laws to define logical equivalences of a statement

In this section, we will learn how to construct logical statements. We will later combine our knowledge of sets with what we will learn about constructing logical statements to analyze arguments with logic.

Logic is a systematic way of thinking that allows us to deduce new information from old information and to parse the meanings of sentences. You use logic informally in everyday life and certainly also in doing mathematics. For example, suppose you are working with a certain circle, call it "Circle X," and you have available the following two pieces of information.

- Circle X has radius equal to 3.
- If any circle has radius r , then its area is πr^2 square units.

You have no trouble putting these two facts together to get:

- Circle X has area 9π square units.

You are using logic to combine existing information to produce new information. Since a major objective in mathematics is to deduce new information, logic must play a fundamental role. This chapter is intended to give you a sufficient mastery of logic.

BOOLEAN LOGIC

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like “and” and “or” – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

Boolean Logic

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like “and,” “or,” and “not” to connect our keywords together to form a search. These words, which form the basis of **Boolean logic**, are directly related to set operations with the same terminology.

Boolean Logic

Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.

- In connection to sets, a boolean search is true if the element in question is part of the set being searched.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for “mystery”, we are looking for all the books that are an element of the set M ; the search is true for books that are in the set.

When we search for “mystery *and* comedy” we are looking for a book that is an element of both sets, in the intersection. If we were to search for “mystery *or* comedy” we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for “*not* comedy” we are looking for any book in the library that is not a comedy, the complement of the set C .

Connection to Set Operations

A and B	elements in the intersection $A \cap B$
A or B	elements in the union $A \cup B$
Not A	elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks “do you want to go to the park or the movies?” they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant “would you like fries or a drink with that?” Answering “both, please” is an acceptable answer.

In the following video, You will see examples of how Boolean operators are used to denote sets.

Watch this video online: <https://youtu.be/ZOLinnoXEAw>

Example

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.

Answer

We could start with the search “Mexico *and* university”, but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:
Mexico *and* university *not* “New Mexico”

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google’s search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:

Mexico university -“New Mexico”

Example

Describe the numbers that meet the condition:
even and less than 10 and greater than 0

Answer

The numbers that satisfy all three requirements are {2, 4, 6, 8}

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Which Comes First?

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

The English phrase “Go to the store and buy me eggs and bagels or cereal” is ambiguous; it is not clear whether the requestors is asking for eggs always along with either bagels or cereal, or whether they’re asking for either the combination of eggs and bagels, or just cereal.

For this reason, using parentheses clarifies the intent:

Eggs and (bagels or cereal) means

Option 1: Eggs and bagels, Option 2: Eggs and cereal

(Eggs and bagels) or cereal means

Option 1: Eggs and bagels, Option 2: Cereal

Example

Describe the numbers that meet the condition:

odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

Answer

The first three conditions limit us to the set {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set {3, 5, 9, 15}

Notice that we would have gotten a very different result if we had written

(odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5

The first grouped set of conditions would give {3, 9, 15}. When combined with the last condition, though, this set expands without limits:

{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, ...}

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

Conditionals

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A **statement** is something that is either true or false.

Example

A statement like $3 < 5$ is true; a statement like “a rat is a fish” is false. A statement like “ $x < 5$ ” is true for some values of x and false for others.

When an action is taken or not depending on the value of a statement, it forms a **conditional**.

Statements and Conditionals

A **statement** is either true or false.

A **conditional** is a compound statement of the form

“if p then q ” or “if p then q , else s ”.

In common language, an example of a conditional statement would be “If it is raining, then we’ll go to the mall. Otherwise we’ll go for a hike.”

The statement “If it is raining” is the condition—this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

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Excel

Conditional statements are commonly used in spreadsheet applications like Excel. In Excel, you can enter an expression like

`=IF(A1<2000, A1+1, A1*2)`

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations.

Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying “If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiple A1 by 2”

Example

Write an Excel command that will create the condition “A1 < 3000 and A1 > 100”.

Answer

Enter “AND(A1<3000, A1>100)”. Likewise, for the condition “A1=4 or A1=6” you would enter “OR(A1=4, A1=6)”

Example

Given the Excel expression:

`IF(A1 > 5, 2*A1, 3*A1)`

Find the following:

1. the result if A1 is 3, and
2. the result if A1 is 8

Answer

This is equivalent to saying

If A1 > 5, then calculate 2*A1. Otherwise, calculate 3*A1

If A1 is 3, then the condition is false, since 3 > 5 is not true, so we do the alternate action, and multiple by 3, giving 3*3 = 9

If A1 is 8, then the condition is true, since 8 > 5, so we multiply the value by 2, giving 2*8=16

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Example

An accountant needs to withhold 15% of income for taxes if the income is below \$30,000, and 20% of income if the income is \$30,000 or more. Write an Excel expression that would calculate the amount to withhold.

Answer

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: $0.15 \times \text{income}$. If the income is more than 30,000, we need to calculate 20% of the income: $0.20 \times \text{income}$.

In words we could write "If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20". In Excel, we would write:

`=IF(A1<30000, 0.15*A1, 0.20*A1)`

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As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example

A parent might say to their child "if you clean your room and take out the garbage, then you can have ice cream." Under what circumstances will this conditional be true?

Answer

Here, there are two simpler conditions:

1. The child cleaning her room
2. The child taking out the garbage

Since these conditions were joined with *and*, then the combined conditional will only be true if both simpler conditions are true; if either chore is not completed then the parent's condition is not met.

Notice that if the parent had said “if you clean your room *or* take out the garbage, then you can have ice cream”, then the child would only need to complete one chore to meet the condition.

Example

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents. A certain tax credit applies if someone with no dependents earns less than \$10,000 and has no dependents, or if someone with dependents earns less than \$20,000. Write a rule that describes this.

Answer

There are two ways the rule is met:

income is less than 10,000 *and* dependents is 0, *or*
income is less than 20,000 *and* dependents is not 0.

Informally, we could write these as

$(A1 < 10000 \text{ and } A2 = 0) \text{ or } (A1 < 20000 \text{ and } A2 > 0)$

Notice that the $A2 > 0$ condition is actually redundant and not necessary, since we'd only be considering that *or* case if the first pair of conditions were not met. So this could be simplified to

$(A1 < 10000 \text{ and } A2 = 0) \text{ or } (A1 < 20000)$

In Excel's format, we'd write

= IF (OR(AND(A1 < 10000, A2 = 0), A1 < 20000), “you qualify”, “you don't qualify”)

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TRUTH TABLES

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to break the complex statement into simple statements, and determine whether they are true or false. A table will help keep track of all the truth values of the simple statements that make up a complex statement, leading to an analysis of the full statement.

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example

Suppose you're picking out a new couch, and your significant other says “get a sectional *or* something with a chaise.” Construct a truth table that describes the elements of the conditions of this statement and whether the conditions are met.

Answer

This is a complex statement made of two simpler conditions: “is a sectional,” and “has a chaise.” For simplicity, let’s use S to designate “is a sectional,” and C to designate “has a chaise.” The condition S is true if the couch is a sectional.

A truth table for this would look like this:

S	C	$S \text{ or } C$
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if S is true and C is also true, then the complex statement “ S or C ” is true. This would be a sectional that also has a chaise, which meets our desire.

Remember also that *or* in logic is not exclusive; if the couch has both features, it does meet the condition.

Some symbols that are commonly used for *and*, *or*, and *not* make using a truth table easier.

Symbols

The symbol \wedge is used for *and*: A and B is notated $A \wedge B$.

The symbol \vee is used for *or*: A or B is notated $A \vee B$

The symbol \sim is used for *not*: not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection.

$A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$.

In the previous example, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic Truth Tables

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
-----	-----	------------

T	T	T
T	F	T
F	T	T
F	F	F

A	$\sim A$
T	F
F	T

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Truth tables really become useful when analyzing more complex Boolean statements.

Example

Create a truth table for the statement $A \wedge \sim (B \vee C)$

Answer

It helps to work from the inside out when creating truth tables, and create tables for intermediate operations. We start by listing all the possible truth value combinations for A , B , and C . Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates. This pattern ensures that all combinations are considered. Along with those initial values, we'll list the truth values for the innermost expression, $B \vee C$.

A	B	C	$B \vee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created.

A	B	C	$B \vee C$	$\sim (B \vee C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of A and $\sim (B \vee C)$

A	B	C	$B \vee C$	$\sim (B \vee C)$	$A \wedge \sim (B \vee C)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is only true in one case: if A is true, B is false, and C is false.

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When we discussed conditions earlier, we discussed the type where we take an action based on the value of the condition. We are now going to talk about a more general version of a conditional, sometimes called an **implication**.

Implications

Implications are logical conditional sentences stating that a statement p , called the antecedent, implies a consequence q .
Implications are commonly written as $p \rightarrow q$

Implications are similar to the conditional statements we looked at earlier; $p \rightarrow q$ is typically written as “if p then q ,” or “ p therefore q .” The difference between implications and conditionals is that conditionals we discussed earlier suggest an action—if the condition is true, then we take some action as a result. Implications are a logical statement that suggest that the consequence must logically follow if the antecedent is true.

Example

The English statement “If it is raining, then there are clouds in the sky” is a logical implication. Is this a valid argument, why or why not?

Answer

It is a valid argument because if the antecedent “it is raining” is true, then the consequence “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining. If the antecedent is false, then the implication becomes irrelevant.

Example

A friend tells you that “if you upload that picture to Facebook, you’ll lose your job.” Describe the possible outcomes related to this statement, and determine whether your friend’s statement is invalid.

Answer

There are four possible outcomes:

1. You upload the picture and keep your job.
2. You upload the picture and lose your job.
3. You don’t upload the picture and keep your job.
4. You don’t upload the picture and lose your job.

There is only one possible case where your friend was lying—the first option where you upload the picture and keep your job. In the last two cases, your friend didn’t say anything about what would happen if you didn’t upload the picture, so you can’t conclude their statement is invalid, even if you didn’t upload the picture and still lost your job.

In traditional logic, an implication is considered valid (true) as long as there are no cases in which the antecedent is true and the consequence is false. It is important to keep in mind that symbolic logic cannot capture all the intricacies of the English language.

Truth Values for Implications

p	q	$p \rightarrow q$
-----	-----	-------------------

T	T	T
T	F	F
F	T	T
F	F	T

Example

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

Answer

We start by constructing a truth table for the antecedent.

m	p	$\sim p$	$m \wedge \sim p$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Now we can build the truth table for the implication

m	p	$\sim p$	$m \wedge \sim p$	r	$(m \wedge \sim p) \rightarrow r$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	T	T
T	T	F	F	F	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	F	F	T

In this case, when m is true, p is false, and r is false, then the antecedent $m \wedge \sim p$ will be true but the consequence false, resulting in a invalid implication; every other case gives a valid implication.

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For any implication, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original implication is “if p then q ”: $p \rightarrow q$

The converse is “if q then p ”: $q \rightarrow p$

The inverse is “if not p then not q ”: $\sim p \rightarrow \sim q$

The contrapositive is “if not q then not p ”: $\sim q \rightarrow \sim p$

Example

Consider again the valid implication “If it is raining, then there are clouds in the sky.”
Write the related converse, inverse, and contrapositive statements.

Answer

The converse would be “If there are clouds in the sky, it is raining.” This is certainly not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is valid, and is equivalent to the original implication.

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Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Implication	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalence

A conditional statement and its contrapositive are logically equivalent.
The converse and inverse of a statement are logically equivalent.

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DEMORGAN'S LAWS

There are two pairs of logically equivalent statements that come up again and again in logic. They are prevalent enough to be dignified by a special name: **DeMorgan's laws**.

The laws are named after Augustus De Morgan (1806–1871), who introduced a formal version of the laws to classical propositional logic. De Morgan's formulation was influenced by algebraization of logic undertaken by George Boole, which later cemented De Morgan's claim to the find. Nevertheless, a similar observation was made by Aristotle, and was known to Greek and Medieval logicians. For example, in the 14th century, William of Ockham wrote down the words that would result by reading the laws out. Jean Buridan, in his *Summulae de Dialectica*, also describes rules of conversion that follow the lines of De Morgan's laws. Still, De Morgan is given credit for stating the laws in the terms of modern formal logic, and incorporating them into the language of logic. De Morgan's laws can be proved easily, and may even seem trivial. Nonetheless, these laws are helpful in making valid inferences in proofs and deductive arguments.

DeMorgan's Laws

1. $\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$
2. $\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$

The first of DeMorgan's laws is verified by the following table. You are asked to verify the second in an exercise.

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim (P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

DeMorgan's laws are actually very natural and intuitive. Consider the statement $\sim (P \wedge Q)$, which we can interpret as meaning that it is not the case that both P and Q are true. If it is not the case that both P and Q are true, then at least one of P or Q is false, in which case $(\sim P) \vee (\sim Q)$ is true. Thus $\sim (P \wedge Q)$ means the same thing as $(\sim P) \vee (\sim Q)$.

DeMorgan's laws can be very useful. Suppose we happen to know that some statement having form $\sim (P \vee Q)$ is true. The second of DeMorgan's laws tells us that $(\sim Q) \wedge (\sim P)$ is also true, hence $\sim P$ and $\sim Q$ are both true as well. Being able to quickly obtain such additional pieces of information can be extremely useful.

Here is a summary of some significant logical equivalences. Those that are not immediately obvious can be verified with a truth table.

Contrapositive law $P \rightarrow Q = (\sim Q) \rightarrow (\sim P)$

DeMorgan's laws $\sim (P \wedge Q) = \sim P \vee \sim Q$
 $\sim (P \vee Q) = \sim P \wedge \sim Q$

Commutative laws $(P \wedge Q) = P \wedge Q$
 $(P \vee Q) = P \vee Q$

Distributive laws $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
 $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

Associative laws $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$
 $P \vee (Q \vee R) = (P \vee Q) \vee R$

Notice how the distributive law $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$ has the same structure as the distributive law $p(q + r) = p \cdot q + p \cdot r$ from algebra. Concerning the associative laws, the fact that $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$ means that the position of the parentheses is irrelevant, and we can write this as $P \wedge Q \wedge R$ without ambiguity. Similarly, we may drop the parentheses in an expression such as $P \vee (Q \vee R)$.

But parentheses are essential when there is a mix of \wedge and \vee , as in $P \vee (Q \wedge R)$. Indeed, $P \vee (Q \wedge R)$ and $P \vee Q \wedge R$ are **not** logically equivalent.

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Negating Statements

Given a statement R , the statement $\sim R$ is called the **negation** of R . If R is a complex statement, then it is often the case that its negation $\sim R$ can be written in a simpler or more useful form. The process of finding this form is called **negating** R . In proving theorems it is often necessary to negate certain statements. We now investigate how to do this.

We have already examined part of this topic. **DeMorgan's laws**

$\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$
 $\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$

(from "Logical Equivalence") can be viewed as rules that tell us how to negate the statements $P \wedge Q$ and $P \vee Q$. Here are some examples that illustrate how DeMorgan's laws are used to negate statements involving "and" or "or."

Example

Consider negating the following statement.
 R : You can solve it by factoring or with the quadratic formula.

Answer

Now, R means (You can solve it by factoring) \vee (You can solve it with Q.F.), which we will denote as $P \vee Q$. The negation of this is $\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$.

Therefore, in words, the negation of R is

$\sim R$: You can't solve it by factoring and you can't solve it with the quadratic formula.

Maybe you can find $\sim R$ without invoking DeMorgan's laws. That is good; you have internalized DeMorgan's laws and are using them unconsciously.

Example

We will negate the following sentence.
 R : The numbers x and y are both odd.

Answer

This statement means $(x \text{ is odd}) \wedge (y \text{ is odd})$, so its negation is

$\sim [(x \text{ is odd}) \wedge (y \text{ is odd})] = \sim (x \text{ is odd}) \vee \sim (y \text{ is odd})$

$(x \text{ is odd}) \wedge (y \text{ is odd}) = (x \text{ is even}) \vee (y \text{ is even})$

Therefore the negation of R can be expressed in the following ways:

$\sim R$: The number x is even or the number y is even.

$\sim R$: At least one of x and y is even.

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INTRODUCTION: ANALYZING ARGUMENTS WITH LOGIC

Learning Objectives

Analyzing Arguments With Logic

- Discern between an inductive argument and a deductive argument
- Evaluate deductive arguments
- Analyze arguments with Venn diagrams and truth tables
- Use logical inference to infer whether a statement is true
- Identify logical fallacies in common language including appeal to ignorance, appeal to authority, appeal to consequence, false dilemma, circular reasoning, post hoc, correlation implies causation, and straw man arguments

In the next section we will use what we have learned about constructing statements to build arguments with logical statements. We will also use more Venn diagrams to evaluate whether an argument is logical, and introduce how to use a truth table to evaluate a logical statement.

ARGUMENTS

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

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Example

The argument “when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example

The argument “every day for the past year, a plane flies over my house at 2pm. A plane will fly over my house every day at 2pm” is a stronger inductive argument, since it is based on a larger set of evidence.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. The commonly known scientific theories, like Newton’s theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence. For gravity, this happened when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes them easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if all the premises are true, and the conclusion follows logically from those premises. In other words, the premises are true, and the conclusion follows necessarily from those premises.

Example

The argument “All cats are mammals and a tiger is a cat, so a tiger is a mammal” is a valid deductive argument.

The premises are:

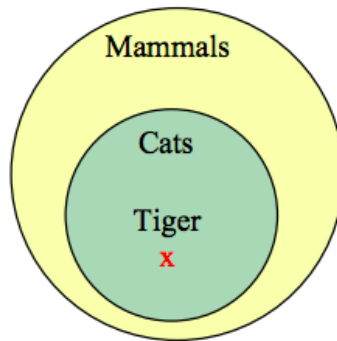
All cats are mammals

A tiger is a cat

The conclusion is:

A tiger is a mammal

Both the premises are true. To see that the premises must logically lead to the conclusion, one approach would be use a Venn diagram. From the first premise, we can conclude that the set of cats is a subset of the set of mammals. From the second premise, we are told that a tiger lies within the set of cats. From that, we can see in the Venn diagram that the tiger also lies inside the set of mammals, so the conclusion is valid.



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Analyzing Arguments with Venn/Euler diagrams

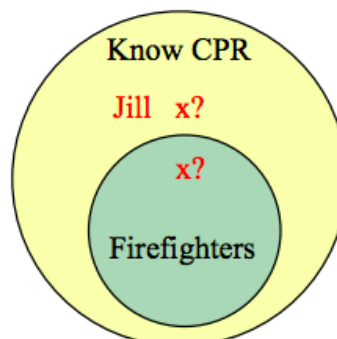
To analyze an argument with a Venn/ Euler diagram

1. Draw a Venn/ Euler diagram based on the premises of the argument
2. If the premises are insufficient to determine what determine the location of an element, indicate that.
3. The argument is valid if it is clear that the conclusion must be true

Example

Premise:	All firefighters know CPR
Premise:	Jill knows CPR
Conclusion:	Jill is a firefighter

From the first premise, we know that firefighters all lie inside the set of those who know CPR. From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know if she also is a member of the smaller subset that is firefighters.



Since the conclusion does not necessarily follow from the premises, this is an invalid argument, regardless of whether Jill actually is a firefighter.

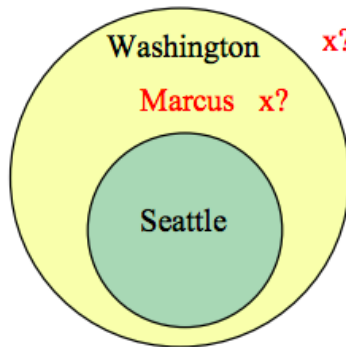
It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are only concerned with whether the premises are enough to prove the conclusion.

In addition to these categorical style premises of the form “all _____,” “some _____,” and “no _____,” it is also common to see premises that are implications.

Example

Premise:	If you live in Seattle, you live in Washington.
Premise:	Marcus does not live in Seattle.
Conclusion:	Marcus does not live in Washington.

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether or not Marcus lives in Washington or not. This is an invalid argument.



Example

Consider the argument “You are a married man, so you must have a wife.”

Answer

This is an invalid argument, since there are, at least in parts of the world, men who are married to other men, so the premise not insufficient to imply the conclusion.

Some arguments are better analyzed using truth tables.

Example

Consider the argument:

Premise:	If you bought bread, then you went to the store
Premise:	You bought bread
Conclusion:	You went to the store

Answer

While this example is hopefully fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then look at the implication that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid. We'll get B represent "you bought bread" and S represent "you went to the store". Then the argument becomes:

Premise:	$B \rightarrow S$
Premise:	B
Conclusion:	S

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(B \rightarrow S) \wedge B] \rightarrow S$?

B	S	$B \rightarrow S$	$(B \rightarrow S) \wedge B$	$[(B \rightarrow S) \wedge B] \rightarrow S$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for $[(B \rightarrow S) \wedge B] \rightarrow S$ is always true, this is a valid argument.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises with and to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for that statement. If it is always true, then the argument is valid.

Example

Premise:	If I go to the mall, then I'll buy new jeans.
----------	---

Premise:	If I buy new jeans, I'll buy a shirt to go with it.
Conclusion:	If I got to the mall, I'll buy a shirt.

Let M = I go to the mall, J = I buy jeans, and S = I buy a shirt.
The premises and conclusion can be stated as:

Premise:	$M \rightarrow J$
Premise:	$J \rightarrow S$
Conclusion:	$M \rightarrow S$

We can construct a truth table for $[(M \rightarrow J) \wedge (J \rightarrow S)] \rightarrow (M \rightarrow S)$

M	J	S	$M \rightarrow J$	$J \rightarrow S$	$(M \rightarrow J) \wedge (J \rightarrow S)$	$M \rightarrow S$	$[(M \rightarrow J) \wedge (J \rightarrow S)] \rightarrow (M \rightarrow S)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the truth table, we can see this is a valid argument.

The previous problem is an example of a syllogism.

Syllogism

A syllogism is an implication derived from two others, where the consequence of one is the antecedent to the other. The general form of a syllogism is:

Premise:	$p \rightarrow q$
Premise:	$q \rightarrow r$
Conclusion:	$p \rightarrow r$

This is sometime called the transitive property for implication.

Example

Premise:	If I work hard, I'll get a raise.
Premise:	If I get a raise, I'll buy a boat.
Conclusion:	If I don't buy a boat, I must not have worked hard.

Answer

If we let W = working hard, R = getting a raise, and B = buying a boat, then we can represent our argument symbolically:

Premise:	$H \rightarrow R$
Premise:	$R \rightarrow B$
Conclusion:	$\sim B \rightarrow \sim H$

We could construct a truth table for this argument, but instead, we will use the notation of the contrapositive we learned earlier to note that the implication $\sim B \rightarrow \sim H$ is equivalent to the implication $H \rightarrow B$. Rewritten, we can see that this conclusion is indeed a logical syllogism derived from the premises.

Try It Now

Is this argument valid?

Premise:	If I go to the party, I'll be really tired tomorrow.
Premise:	If I go to the party, I'll get to see friends.
Conclusion:	If I don't see friends, I won't be tired tomorrow.

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Lewis Carroll, author of *Alice in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms.

Example

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody who can manage a crocodile is despised.

Illogical persons are despised.

Answer

Let B = is a baby, D = is despised, I = is illogical, and M = can manage a crocodile.

Then we can write the premises as:

$$B \rightarrow I$$

$$M \rightarrow \sim D$$

$$I \rightarrow D$$

From the first and third premises, we can conclude that $B \rightarrow D$; that babies are despised.

Using the contrapositive of the second premise, $D \rightarrow \sim M$, we can conclude that $B \rightarrow \sim M$; that babies cannot manage crocodiles.

While silly, this is a logical conclusion from the given premises.

Logical Inference

Suppose we know that a statement of form $P \rightarrow Q$ is true. This tells us that whenever P is true, Q will also be true. By itself, $P \rightarrow Q$ being true does not tell us that either P or Q is true (they could both be false, or P could be false and Q true). However if in addition we happen to know that P is true then it must be that Q is true. This is called a **logical inference**: Given two true statements we can infer that a third statement is true. In this instance true statements $P \rightarrow Q$ and P are “added together” to get Q . This is described below with $P \rightarrow Q$ stacked one atop the other with a line separating them from Q . The intended meaning is that $P \rightarrow Q$ combined with P produces Q .

$\begin{array}{l} P \rightarrow Q \\ P \\ \hline Q \end{array}$	$\begin{array}{l} P \rightarrow Q \\ \sim Q \\ \hline \sim P \end{array}$	$\begin{array}{l} P \vee Q \\ \sim P \\ \hline Q \end{array}$
---	---	---

Two other logical inferences are listed above. In each case you should convince yourself (based on your knowledge of the relevant truth tables) that the truth of the statements above the line forces the statement below the line to be true.

Following are some additional useful logical inferences. The first expresses the obvious fact that if P and Q are both true then the statement $P \wedge Q$ will be true. On the other hand, $P \wedge Q$ being true forces P (also Q) to be true. Finally, if P is true, then $P \vee Q$ must be true, no matter what statement Q is.

$\begin{array}{l} P \\ Q \\ \hline P \wedge Q \end{array}$	$\begin{array}{l} P \wedge Q \\ P \\ \hline P \end{array}$	$\begin{array}{l} P \\ \hline P \vee Q \end{array}$
--	--	---

The first two statements in each case are called “premises” and the final statement is the “conclusion.” We combine premises with \wedge (“and”). The premises together imply the conclusion. Thus, the first argument would have $((P \rightarrow Q) \wedge P) \rightarrow Q$

An Important Note

It is important to be aware of the reasons that we study logic. There are three very significant reasons. First, the truth tables we studied tell us the exact meanings of the words such as “and,” “or,” “not,” and so on. For instance, whenever we use or read the “If..., then” construction in a mathematical context, logic tells us exactly what is meant. Second, the rules of inference provide a system in which we can produce new information (statements) from known information. Finally, logical rules such as DeMorgan’s laws help us correctly change certain statements into (potentially more useful) statements with the same meaning. Thus logic helps us understand the meanings of statements and it also produces new meaningful statements.

Logic is the glue that holds strings of statements together and pins down the exact meaning of certain key phrases such as the “If..., then” or “For all” constructions. Logic is the common language that all mathematicians use, so we must have a firm grip on it in order to write and understand mathematics.

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LOGICAL FALLACIES IN COMMON LANGUAGE

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

Ad hominem

An ad hominem argument attacks the person making the argument, ignoring the argument itself.

Example

“Jane says that whales aren’t fish, but she’s only in the second grade, so she can’t be right.”

Answer

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example

“Jane says that whales aren’t fish, but everyone knows that they’re really mammals—she’s so stupid.”

Answer

This certainly isn’t very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

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Appeal to ignorance

This type of argument assumes something is true because it hasn't been proven false.

Example

"Nobody has proven that photo isn't Bigfoot, so it must be Bigfoot."

Appeal to authority

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person's opinion is not shared by other experts, or when the authority is irrelevant to the claim.

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Example

"A diet high in bacon can be healthy – Doctor Atkins said so."

Answer

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example

"Jennifer Hudson lost weight with Weight Watchers, so their program must work."

Answer

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person's experience would.

Appeal to Consequence

An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example

“Humans will travel faster than light: faster-than-light travel would be beneficial for space travel.”

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False dilemma

A false dilemma argument falsely frames an argument as an “either or” choice, without allowing for additional options.

Example

“Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens.”

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

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Circular reasoning

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example

“I shouldn’t have gotten a C in that class; I’m an A student!”

In this argument, the student is claiming that because they're an A student, though shouldn't have gotten a C. But because they got a C, they're not an A student.

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Straw man

A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example

"Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists"
Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack.

Post hoc (post hoc ergo propter hoc)

A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

Example

"Today I wore a red shirt, and my football team won! I need to wear a red shirt everytime they play to make sure they keep winning."

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Correlation implies causation

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

Example

“Months with high ice cream sales also have a high rate of deaths by drowning. Therefore ice cream must be causing people to drown.”

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

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Try It Now

Identify the logical fallacy in each of the arguments

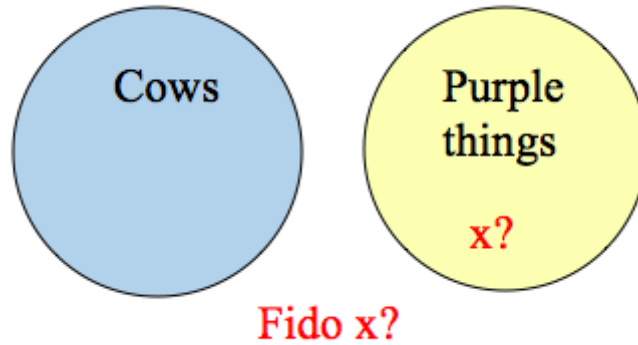
1. Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
2. Since the 1950s, both the atmospheric carbon dioxide level and **obesity** levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
3. The oven was working fine until you started using it, so you must have broken it.
4. You can't give me a D in the class—I can't afford to retake it.
5. The senator wants to increase support for food stamps. He wants to take the taxpayers' hard-earned money and give it away to lazy people. This isn't fair so we shouldn't do it.

Answer

1.

A	B	$\sim A$	$\sim A \wedge B$	$\sim B$	$(\sim A \wedge B) \vee \sim B$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

2. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.



3. Let S : have a shovel, D : dig a hole. The first premise is equivalent to $S \rightarrow D$. The second premise is D . The conclusion is S .

We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$

S	D	$S \rightarrow D$	$(S \rightarrow D) \wedge D$	$[(S \rightarrow D) \wedge D] \rightarrow S$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

This is not a tautology, so this is an invalid argument.

4. Letting P = go to the party, T = being tired, and F = seeing friends, then we can represent this argument as P :

Premise:	$P \rightarrow T$
Premise:	$P \rightarrow F$
Conclusion:	$\sim F \rightarrow \sim T$

We could rewrite the second premise using the contrapositive to state $\sim F \rightarrow \sim P$, but that does not allow us to form a syllogism. If we don't see friends, then we didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.

5.

- Circular
- Correlation does not imply causation
- Post hoc
- Appeal to consequence
- Straw man

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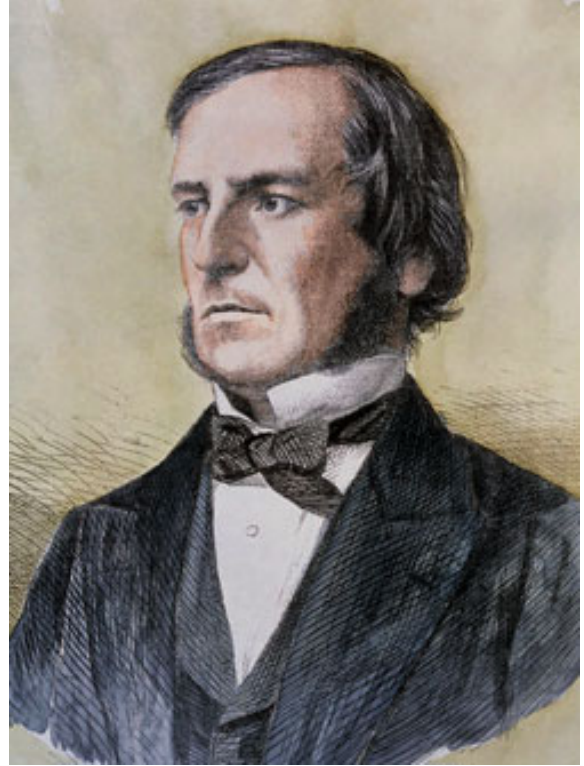
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PUTTING IT TOGETHER: SET THEORY AND LOGIC

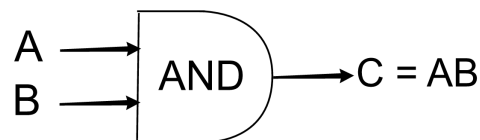
In this module we've seen how logic and valid arguments can be formalized using mathematical notation and a few basic rules. In fact when George Boole (1815-1864) first developed **symbolic logic** (or **Boolean logic**), he had the idea that his system could be used by lawyers, philosophers, and mathematicians alike to help put convoluted arguments on a firmer footing. Little did he realize that his system of "*and*," "*or*," and "*not*" operations would one day transform the world by ushering in the Digital Revolution and modern day computing.

What is the connection between logic and computers? Instead of truth values **T** and **F**, digital computers rely on two **states**, either *on*(1) or *off*(0). This is because a computer consists of many **circuits**, which are electrical pathways that can either be closed to allow the current to flow, or open to break the connection. A "1" would signify a closed circuit while a "0" represents an open circuit.



George Boole

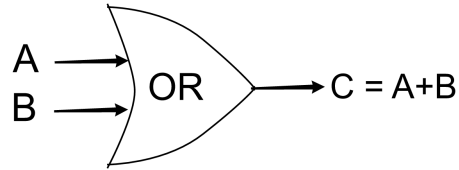
Certain components called **gates** allow the computer to open or close circuits based on input. For example, an AND gate has two input wires (A, B) and one output (C). Electricity will flow at C if and only if both A and B have current. Traditionally, the AND operation is written like multiplication; that is, $A \text{ AND } B = AB$.



Multiplication seems to be a natural interpretation of AND when applied to the values 0 and 1. Just think about the truth table for the operation \wedge , replacing **T** by 1 and **F** by 0.

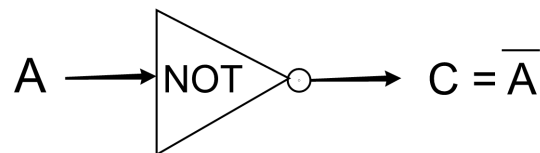
A	B	AB (A AND B)
1	1	1
1	0	0
0	1	0
0	0	0

There is also an OR gate. Again, two inputs A and B determine the output C, however this time $C = 1$ if and only if either A or B (or both) is equal to 1. This operation, which corresponds to the logical expression $A \vee B$, is often interpreted as a kind of addition ($A \text{ OR } B = A + B$), however it's not a perfect analogy because $1 + 1 = 1$ in Boolean logic.



A	B	A + B (A OR B)
1	1	1
1	0	1
0	1	1
0	0	0

Finally, there is a gate whose output is the opposite state as its input. So if the input (A) is 1, then the output (C) will be 0, and vice versa. This is called the NOT gate. You have encountered “not” as the logical expression $\sim A$, but the usual notation in computer science for NOT A is \overline{A} . The gate along with its truth table shown below.

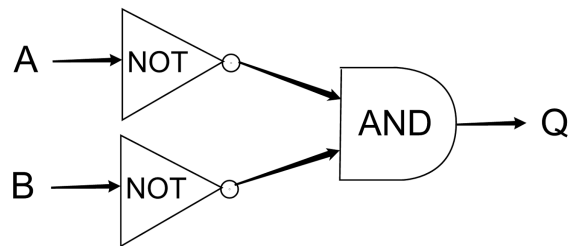


A	\overline{A} (NOT A)
1	0
0	1

Moreover, numerical values can be represented by a string of 1's and 0's in what we call **binary notation**.

Then the basic operations of addition, subtraction, multiplication, and division of binary number can actually be accomplished using the right combination of gates, in other words by Boolean logical operations.

However a proper discussion of binary arithmetical falls outside the scope of this discussion. Instead, let's use Boolean logic and to find a simpler circuit equivalent to the one shown.



The given circuit has three gates. Can you find a circuit with only two gates that produces exactly the same output (Q) for all choices of input (A, B)?

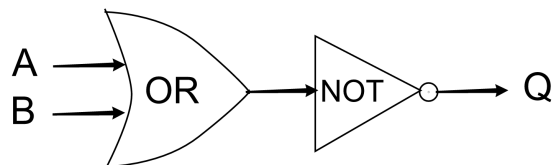
Let's translate the diagram into a Boolean expression. First, both A and B are negated to obtain \overline{A} and \overline{B} , respectively. Those expressions in turn feed into the AND gate. So $Q = \overline{A} \cdot \overline{B}$. In terms of the logical operations you have studied in this module,

$$Q = \overline{A} \cdot \overline{B} = (\sim A) \wedge (\sim B)$$

You may recognize the expression as one side of De Morgan's Law. Therefore, there is an equivalence,

$$(\sim A) \wedge (\sim B) = \sim (A \vee B) = \overline{A + B}$$

Finally, the last expression corresponds to a circuit diagram with only two gates, an OR and a NOT.



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VOTING THEORY

WHY IT MATTERS: VOTING THEORY

Why study voting theory?

Every four years in the United States, there is a major election in which citizens over the age of 18 cast their ballots for the president, vice president, and various other offices of the government.

Elections for state and local leaders as well as some federal legislators occur every two years, or even yearly.

Because our choices for who fills these seats will affect our lives for years to come, it is very important to understand how the voting process works, as well as the theory that ensures a level of fairness in the outcome.



United States President Barack Obama casts his ballot during early voting in the 2012 U.S. election at the Martin Luther King Jr. Community Center in Chicago, Illinois on 25 October 2012, making him the first U.S. President to vote early.

We may live in a democracy, but contrary to popular thought, the president is not directly elected by the people.

Instead, in the **general election** in November, our votes go to **electors**, who are pledged to vote for a particular candidate. The electors form a body called the **electoral college** that ultimately decide the presidential election about a month after the popular vote has concluded.

Rules concerning how the electoral college works vary from state to state. In most states, whichever candidate receives the most votes automatically receives all of the electoral college votes in that state, a system

commonly known as **winner take all**. In other states, like Nebraska and Maine, the electoral college vote can split by district.

But let's back up, because the presidential election actually begins in the Spring and Summer before the general election with the **primary election**. In the primary election, voters also cast ballots for their preferred candidate for president of the United States of America. In effect we vote twice! The difference is that the primary typically serves to narrow down the field of candidates within a particular party. Registered Republicans get to choose from a slate of Republican presidential candidates, and Democrats do the same among Democratic candidates. In this way, as the theory goes, each party can choose the candidate that best represents its own unique values and governing policy.

But not every state holds a primary election. Some hold **caucuses** instead. Basically, a caucus is a meeting of a small number of party members that discuss and debate the candidates' merits and then choose amongst themselves which one might best represent the party in that state. Regardless of whether a state uses a primary or a caucus system (rules can differ by party even within the same state), each state's party officials send delegates to a national convention to cast their votes for their candidate of choice. At the convention, the party's candidate is officially nominated and can look forward to the November general election.

So why is our presidential election system so complicated? For one thing, the system was not set up in one day, but evolved over the years due to pressures from various groups. If certain problems were observed in a given year, then rules might be introduced to prevent that problem in future elections. Often the new rules would cause new kinds of problems to creep into the system, necessitating further rules. Some procedures were put in place because of a deep commitment to fairness and democracy for all, while other rules may have been politically motivated, giving one party or faction an artificial advantage or disadvantage. Even if our system is not perfect, we can be confident that it is not done changing. Every election cycle seems to highlight different issues for concern, and we keep doing our best to fix it.

In this module, you will learn about a number of basic voting systems, including plurality, Borda count, approval method and others. Each one has its own pros and cons. Indeed, there is no perfect voting system, a result known as **Arrow's Impossibility Theorem**.

Learning Objectives

Preference Ballot Voting

- Given the results of a preference ballot, determine the winner of an election using the plurality method
- Identify flaws in the plurality voting method
- Identify situations that may lead to insincere voting
- Given the results of a preference ballot, determine the winner of an election using the instant runoff voting method
- Identify situations when the instant runoff voting method produces a violation of the Condorcet Winner
- Given the results of a preference ballot, determine the winner of an election using the Borda Count
- Identify situations where the Borda count violates the fairness criterion
- Given the results of a preference ballot, determine the winner of an election using Copeland's method
- Identify situations where Copeland's method violates the independence of irrelevant alternatives criterion

Approval Voting

- Given the results of an approval ballot determine the winner of an election
- Identify how approval voting can violate the majority criterion

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INTRODUCTION: PREFERENCE BALLOT VOTING

Learning Objectives

- Determine the winner of an election using preference ballots
- Evaluate the fairness of an election using preference ballots
- Determine the winner of an election using the Instant Runoff method
- Evaluate the fairness of an Instant Runoff election
- Determine the winner of an election using a Borda count
- Evaluate the fairness of an election determined using a Borda count
- Determine the winner of an election using Copeland's method
- Evaluate the fairness of an election determined by Copeland's method

In many decision making situations, it is necessary to gather the group consensus. This happens when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leaders.

While the basic idea of voting is fairly universal, the method by which those votes are used to determine a winner can vary. Amongst a group of friends, you may decide upon a movie by voting for all the movies you're willing to watch, with the winner being the one with the greatest approval. A company might eliminate unpopular designs then revote on the remaining. A country might look for the candidate with the most votes.

In deciding upon a winner, there is always one main goal: to reflect the preferences of the people in the most fair way possible.

In this lesson, we will study Preference Schedules and Preference Ballots as a means of deciding upon the winner of an election. A **preference ballot** is a ballot in which the voter ranks the choices in order of preference.

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PLURALITY METHOD

Preference Schedules

To begin, we're going to want more information than a traditional ballot normally provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice was unsuccessful.

Preference ballot

A **preference ballot** is a ballot in which the voter ranks the choices in order of preference.

Example

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Their votes are shown below:

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
1st choice	A	A	O	H	A	O	H	O	H	A
2nd choice	O	H	H	A	H	H	A	H	A	H
3rd choice	H	O	A	O	O	A	O	A	O	O

These individual ballots are typically combined into one **preference schedule**, which shows the number of voters in the top row that voted for each option:

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Notice that by totaling the vote counts across the top of the preference schedule we can recover the total number of votes cast: $1 + 3 + 3 + 3 = 10$ total votes.

The following video will give you a summary of what issues can arise from elections, as well as how a preference table is used in elections.

Watch this video online: <https://youtu.be/6rhpq1ozmuQ>

Plurality

The voting method we're most familiar with in the United States is the **plurality method**.

Plurality Method

In this method, the choice with the most first-preference votes is declared the winner. Ties are possible, and would have to be settled through some sort of run-off vote.

This method is sometimes mistakenly called the majority method, or "majority rules", but it is not necessary for a choice to have gained a majority of votes to win. A majority is over 50%; it is possible for a winner to have a **plurality** without having a majority.

Example

In our election from above, we had the preference table:

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

For the plurality method, we only care about the first choice options. Totaling them up:

Anaheim: $1 + 3 = 4$ first-choice votes

Orlando: 3 first-choice votes

Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes, but not a majority.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Try It Now

Three candidates are running in an election for County Executive: Goings (G), McCarthy (M), and Bunney (B) (Note: This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See <http://www.co.pierce.wa.us/xml/abtus/ourorg/aud/Elections/RCV/ranked/exec/summary.pdf>) The voting schedule is shown below. Which candidate wins under the plurality method?

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	

3rd choice	B	M		B	G	G	
------------	---	---	--	---	---	---	--

Note: In the third column and last column, those voters only recorded a first-place vote, so we don't know who their second and third choices would have been.

What's Wrong with Plurality?

The election from the above example may seem totally clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices?

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

This doesn't seem right, does it? Anaheim just won the election, yet 6 out of 10 voters, 60% of them, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist wrote about how this could happen in 1785, and for him we name our first **fairness criterion**.

Fairness Criteria

The fairness criteria are statements that seem like they should be true in a fair election.

Condorcet Criterion

If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the **Condorcet Winner**, or Condorcet Candidate.

Example

In the election, what choice is the Condorcet Winner?

We see above that Hawaii is preferred over Anaheim. Comparing Hawaii to Orlando, we can see 6 out of 10 would prefer Hawaii to Orlando.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Since Hawaii is preferred in a one-to-one comparison to both other choices, Hawaii is the Condorcet Winner.

Example

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:

	342	214	298
1st choice	Elle	Don	Key
2nd choice	Don	Key	Don
3rd choice	Key	Elle	Elle

We can see a total of $342 + 214 + 298 = 854$ voters participated in this election. Computing percentage of first place votes:

Don: $214/854 = 25.1\%$

Key: $298/854 = 34.9\%$

Elle: $342/854 = 40.0\%$

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closer, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 214 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 298 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

So even though Don had the smallest number of first-place votes in the election, he is the Condorcet winner, being preferred in every one-to-one comparison with the other candidates.

If you prefer to watch a video of the previous example being worked out, here it is.

Watch this video online: <https://youtu.be/x6DpoeaRVsw>

Try It Now

Consider the election from the previous Try It Now. Is there a Condorcet winner in this election?

	44	14	20	70	22	80	39
--	----	----	----	----	----	----	----

1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Insincere Voting

Situations when there are more than one candidate that share somewhat similar points of view, can lead to **insincere voting**. Insincere voting is when a person casts a ballot counter to their actual preference for strategic purposes. In the case above, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

The following video gives another mini lesson that covers the plurality method of voting as well as the idea of a Condorcet Winner.

Watch this video online: <https://youtu.be/r-VmxJQFMq8>

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INSTANT RUNOFF VOTING

Instant Runoff Voting

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting. In IRV, voting is done with preference ballots, and a preference schedule is generated. The choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%).

This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans, called A, B, C, D, and E here for simplicity.

Initial votes

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

If this was a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are total of $3+4+4+6+2+1 = 20$ votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Round 1: We make our first elimination. Choice A has the fewest first-place votes, so we remove that choice

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C		D	C	E	
3rd choice		D	C			D
4th choice	D	B		E	C	B
5th choice	E	E	E	B	D	C

We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	D	D	C	E	D
3rd choice	D	B	C	E	C	B
4th choice	E	E	E	B	D	C

Round 2: We make our second elimination. Choice E has the fewest first-place votes, so we remove that choice, shifting everyone's options to fill the gaps.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	D
2nd choice	C	D	D	C	C	B
3rd choice	D	B	C	B	D	C

Notice that the first and fifth columns have the same preferences now, we can condense those down to one column.

	5	4	4	6	1
1st choice	B	C	B	D	D
2nd choice	C	D	D	C	B
3rd choice	D	B	C	B	C

Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes.

	5	4	4	6	1
1st choice	B	D	B	D	D
2nd choice	D	B	D	B	B

Condensing this down:

	9	11
1st choice	B	D
2nd choice	D	B

D has now gained a majority, and is declared the winner under IRV.

The following video provides another view of the example from above.

Watch this video online: https://youtu.be/C-X-6Lo_xUQ

Try It Now

Consider again this election. Find the winner using IRV.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Here is an overview video that provides the definition of IRV, as well as an example of how to determine the winner of an election using IRV.

Watch this video online: <https://youtu.be/6axH6pcuyhQ>

What's Wrong with IRV?

Example

Let's return to our City Council Election.

	342	214	298
1st choice	Elle	Don	Key
2nd choice	Don	Key	Don
3rd choice	Key	Elle	Elle

In this election, Don has the smallest number of first place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.

	342	512
1st choice	Elle	Key
2nd choice	Key	Elle

So Key is the winner under the IRV method.

We can immediately notice that in this election, IRV violates the Condorcet Criterion, since we determined earlier that Don was the Condorcet winner. On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

In the following video, we provide the example from above where we find that the IRV method violates the Condorcet Criterion in an election for a city council seat.

Watch this video online: <https://youtu.be/BCRaYCU28Ro>

Example

Consider the voting system below.

	37	22	12	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

In this election, Carter would be eliminated in the first round, and Adams would be the winner with 66 votes to 34 for Brown.

Now suppose that the results were announced, but election officials accidentally destroyed the ballots before they could be certified, and the votes had to be recast. Wanting to "jump on the bandwagon," 10 of the voters who had originally voted in the order Brown, Adams, Carter change their vote to favor the presumed winner, changing those votes to Adams, Brown, Carter.

	47	22	2	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

In this re-vote, Brown will be eliminated in the first round, having the fewest first-place votes. After transferring votes, we find that Carter will win this election with 51 votes to Adams' 49 votes! Even though the only vote changes made *favored* Adams, the change ended up costing Adams the election. This doesn't seem right, and introduces our second fairness criterion:

Monotonicity Criterion

If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.

This criterion is violated by this election. Note that even though the criterion is violated in this particular election, it does not mean that IRV always violates the criterion; just that IRV has the potential to violate the criterion in certain elections.

The last video shows the example from above where the monotonicity criterion is violated.

Watch this video online: <https://youtu.be/NH78zNXHKUs>

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BORDA COUNT

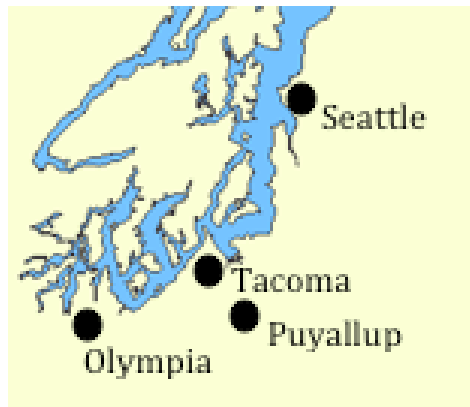
Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

Borda Count

In this method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Example

A group of mathematicians are getting together for a conference. The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown below.



The votes for where to hold the conference were:

	51	25	10	14
1st choice	Seattle	Tacoma	Puyallup	Olympia
2nd choice	Tacoma	Puyallup	Tacoma	Tacoma
3rd choice	Olympia	Olympia	Olympia	Puyallup
4th choice	Puyallup	Seattle	Seattle	Seattle

Use the Borda count method to determine the winning town for the conference.

Answer

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate points awarded

	51	25	10	14
1st choice 4 points	Seattle $4 \cdot 51 = 204$	Tacoma $4 \cdot 25 = 100$	Puyallup $4 \cdot 10 = 40$	Olympia $4 \cdot 14 = 56$
2nd choice 3 points	Tacoma $3 \cdot 51 = 153$	Puyallup $3 \cdot 25 = 75$	Tacoma $3 \cdot 10 = 30$	Tacoma $3 \cdot 14 = 42$
3rd choice 2 points	Olympia $2 \cdot 51 = 102$	Olympia $2 \cdot 25 = 50$	Olympia $2 \cdot 10 = 20$	Puyallup $2 \cdot 14 = 28$
4th choice 1 point	Puyallup $1 \cdot 51 = 51$	Seattle $1 \cdot 25 = 25$	Seattle $1 \cdot 10 = 10$	Seattle $1 \cdot 14 = 14$

Adding up the points:

Seattle: $204 + 25 + 10 + 14 = 253$ points

Tacoma: $153 + 100 + 30 + 42 = 325$ points

Puyallup: $51 + 75 + 40 + 28 = 194$ points

Olympia: $102 + 50 + 20 + 56 = 228$ points

Under the Borda Count method, Tacoma is the winner of this vote.

Here is a video showing the example from above.

Watch this video online: https://youtu.be/vfujywLdW_s

Try It Now

Consider again the election from earlier. Find the winner using Borda Count. Since we have some incomplete preference ballots, for simplicity, give every unranked candidate 1 point, the points they would normally get for last place.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

What's Wrong with Borda Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd, and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from the previous example using the Borda Count violates the Majority Criterion. Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system, since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is a different approach than plurality and instant runoff voting that focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation of it, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

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COPELAND'S METHOD

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

Copeland's Method

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $\frac{1}{2}$ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Example

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Answer

We need to look at each pair of choices, and see which choice would win in a one-to-one comparison. You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice—we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

	1	3	3	3
1st choice			O	H

2nd choice	O	H	H	
3rd choice	H	O		O

Based on this, in the comparison of Hawaii vs Orlando, Hawaii wins, and receives 1 point. Comparing Anaheim to Orlando, the 1 voter in the first column clearly prefers Anaheim, as do the 3 voters in the second column. The 3 voters in the third column clearly prefer Orlando. The 3 voters in the last column prefer Hawaii as their first choice, but if they had to choose between Anaheim and Orlando, they'd choose Anaheim, their second choice overall. So, altogether $1 + 3 + 3 = 7$ voters prefer Anaheim over Orlando, and 3 prefer Orlando over Anaheim. So, comparing Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point.

All together,
Hawaii vs Orlando: 6 votes to 4 votes: Hawaii gets 1 point
Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point
Hawaii vs Anaheim: 6 votes to 4 votes: Hawaii gets 1 point
Hawaii is the winner under Copeland's Method, having earned the most points.
Notice this process is consistent with our determination of a Condorcet Winner.

Here is the same example presented in a video.

Watch this video online: <https://youtu.be/FftVPk7dqV0>

Example

Consider the advertising group's vote we explored earlier. Determine the winner using Copeland's method.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

Answer

With 5 candidates, there are 10 comparisons to make:

A vs B: 11 votes to 9 votes	A gets 1 point
A vs C: 3 votes to 17 votes	C gets 1 point
A vs D: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, D gets $\frac{1}{2}$ point
A vs E: 17 votes to 3 votes	A gets 1 point
B vs C: 10 votes to 10 votes	B gets $\frac{1}{2}$ point, C gets $\frac{1}{2}$ point
B vs D: 9 votes to 11 votes	D gets 1 point

B vs E: 13 votes to 7 votes	B gets 1 point
C vs D: 9 votes to 11 votes	D gets 1 point
C vs E: 17 votes to 3 votes	C gets 1 point
D vs E: 17 votes to 3 votes	D gets 1 point

Totaling these up:

A gets $2\frac{1}{2}$ points

B gets $1\frac{1}{2}$ points

C gets $2\frac{1}{2}$ points

D gets $3\frac{1}{2}$ points

E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

Watch the same example from above being worked out in this video.

Watch this video online: https://youtu.be/sWdmkee5m_Q

Try It Now

Consider again the election from earlier. Find the winner using Copeland's method. Since we have some incomplete preference ballots, we'll have to adjust. For example, when comparing M to B, we'll ignore the 20 votes in the third column which do not rank either candidate.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

What's Wrong with Copeland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, no.

Example

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:

	5	5	6	4
1st choice	D	A	C	B

2nd choice	A	C	B	D
3rd choice	C	B	D	A
4th choice	B	D	A	C

Making the comparisons:

A vs B: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes:	A gets 1 point
A vs D: 5 votes to 15 votes:	D gets 1 point
B vs C: 4 votes to 16 votes:	C gets 1 point
B vs D: 15 votes to 5 votes:	B gets 1 point
C vs D: 11 votes to 9 votes:	C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points	B has $1\frac{1}{2}$ points
C has 2 points	D has 1 point

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:

	5	5	6	4
1st choice	A	A	C	B
2nd choice	C	C	B	A
3rd choice	B	B	A	C

A vs B: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes	A gets 1 point
B vs C: 4 votes to 16 votes	C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points	B has $\frac{1}{2}$ point
C has 1 point	

Suddenly Anna is the winner! This leads us to another fairness criterion.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the winner of the election. Equivalently, if choice A is preferred over choice B, introducing or removing a choice C should not cause B to be preferred over A.

In the election from the last example, the IIA Criterion was violated.

Watch this video to see the example from above worked out again,

Watch this video online: <https://youtu.be/463jDBNR-qY>

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser:

After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says “In that case I’ll have the blueberry pie.”

Another disadvantage of Copeland’s Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland’s method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is not a Condorcet Candidate.

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WHICH METHOD IS FAIR?

So Where’s the Fair Method?

At this point, you’re probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method, right?

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow’s Impossibility Theorem

Arrow’s Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criteria that we’ve discussed.

To see a very simple example of how difficult voting can be, consider the election below:

	5	5	5
1st choice	A	C	B
2nd choice	B	A	C
3rd choice	C	B	A

Notice that in this election:

10 people prefer A to B

10 people prefer B to C

10 people prefer C to A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox**, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

INTRODUCTION: APPROVAL VOTING

Learning Objectives

- Conduct internet searches and write statements using Boolean logic
- Construct a truth table
- Use DeMorgan's laws to write and negate logical statements

Up until now, we've been considering voting methods that require ranking of candidates on a preference ballot. There is another method of voting that can be more appropriate in some decision making scenarios.

In this lesson we will study the method of Approval Voting, its flaws, and how voting is done in the US.

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APPROVAL VOTING

With Approval Voting, the ballot asks you to mark all choices that you find acceptable. The results are tallied, and the option with the most approval is the winner.

Example

A group of friends is trying to decide upon a movie to watch. Three choices are provided, and each person is asked to mark with an “X” which movies they are willing to watch. The results are:

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix	X	X	X	X	X		X			X

Totaling the results, we find:

Titanic received 5 approvals

Scream received 6 approvals

The Matrix received 7 approvals.

In this vote, The Matrix would be the winner.

In the following video you will see the example from above.

Watch this video online: <https://youtu.be/-8put6XKw20>

Try It Now

Our mathematicians deciding on a conference location from earlier decide to use Approval voting. Their votes are tallied below. Find the winner using Approval voting.

	30	10	15	20	15	5	5
Seattle	X	X	X			X	
Tacoma	X		X	X	X	X	
Puyallup		X		X	X	X	
Olympia			X		X		X

What's Wrong with Approval Voting?

Approval voting can very easily violate the Majority Criterion.

Example

Consider the voting schedule:

	80	15	5
1st choice	A	B	C
2nd choice	B	C	B
3rd choice	C	A	A

Clearly A is the majority winner. Now suppose that this election was held using Approval Voting, and every voter marked approval of their top two candidates.
A would receive approval from 80 voters
B would receive approval from 100 voters
C would receive approval from 20 voters
B would be the winner. Some argue that Approval Voting tends to vote the least disliked choice, rather than the most liked candidate.

Additionally, Approval Voting is susceptible to strategic insincere voting, in which a voter does not vote their true preference to try to increase the chances of their choice winning. For example, in the movie example above, suppose Bob and Alice would much rather watch *Scream*. They remove *The Matrix* from their approval list, resulting in a different result.

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix		X	X		X		X			X

Totaling the results, we find *Titanic* received 5 approvals, *Scream* received 6 approvals, and *The Matrix* received 5 approvals. By voting insincerely, Bob and Alice were able to sway the result in favor of their preference.

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VOTING IN AMERICA

In American politics, there is a lot more to selecting our representatives than simply casting and counting ballots. The process of selecting the president is even more complicated, so we'll save that for the next chapter. Instead, let's look at the process by which state congressional representatives and local politicians get elected.

For most offices, a sequence of two public votes is held: a primary election and the general election. For non-partisan offices like sheriff and judge, in which political party affiliation is not declared, the primary election is usually used to narrow the field of candidates.

Typically, the two candidates receiving the most votes in the primary will then move forward to the general election. While somewhat similar to instant runoff voting, this is actually an example of **sequential voting**—a process in which voters cast totally new ballots after each round of eliminations. Sequential voting has become quite common in television, where it is used in reality competition shows like *American Idol*.

Congressional, county, and city representatives are partisan offices, in which candidates usually declare themselves a member of a political party, like the Democrats, Republicans, the Green Party, or one of the many other smaller parties. As with non-partisan offices, a primary election is usually held to narrow down the field prior to the general election. Prior to the primary election, the candidate would have met with the political party leaders and gotten their approval to run under that party's affiliation.

In some states a **closed primary** is used, in which only voters who are members of the Democrat party can vote on the Democratic candidates, and similar for Republican voters. In other states, an **open primary** is used, in which any voter can pick the party whose primary they want to vote in. In other states, **caucuses** are used, which are basically meetings of the political parties, only open to party members. Closed primaries are often disliked by independent voters, who like the flexibility to change which party they are voting in. Open primaries do have the disadvantage that they allow raiding, in which a voter will vote in their non-preferred party's primary with the intent of selecting a weaker opponent for their preferred party's candidate.

Washington State currently uses a different method, called a **top 2 primary**, in which voters select from the candidates from all political parties on the primary, and the top two candidates, regardless of party affiliation, move on to the general election. While this method is liked by independent voters, it gives the political parties incentive to select a top candidate internally before the primary, so that two candidates will not split the party's vote.

Regardless of the primary type, the general election is the main election, open to all voters. Except in the case of the top 2 primary, the top candidate from each major political party would be included in the general election. While rules vary state-to-state, for an independent or minor party candidate to get listed on the ballot, they typically have to gather a certain number of signatures to petition for inclusion.

How can someone win a US Presidential election, even if they don't win the popular vote?

In the 2016 US presidential election, Donald Trump won 306 electoral college votes compared to Hillary Clinton's 232, yet Hillary's tally for the popular vote was 65,794,399 while Trump's was 62,955,202. How does this work?

According to the Pew Research Center (Note: <http://www.pewresearch.org/fact-tank/2016/12/20/why-electoral-college-landslides-are-easier-to-win-than-popular-vote-ones/>), this happened because Trump won large states such as Florida, Pennsylvania, and Wisconsin by very narrow margins. This allowed him to gain all the electoral college votes available for those states.

How does the electoral college work? This ~5 minute video, provides a brief, and interesting introduction to how a citizen's vote is translated into an electoral college vote (or how it is supposed to be).

Watch this video online: <https://youtu.be/OUS9mM8Xbbw>

Scary, huh? In fact, according to the Pew Research Center,

In the vast majority of U.S. elections, in which the same candidate won both the popular and the electoral vote, the system usually makes the winner's victory margin in the former a lot wider than in the latter. In 2012, for example, Barack Obama won 51% of the nationwide popular vote but nearly 62% of the electoral votes, or 332 out of 538.

Looking back at all presidential elections since 1828, the winner's electoral vote share has, on average, been 1.36 times his popular vote share—what we'll call the electoral vote (EV) inflation factor. Trump's EV inflation factor, based on his winning 56.5% of the electoral votes (304 out of 538) is 1.22, similar to Obama's in 2012 (1.21).

Since all but two states use a plurality system, recall that this means the winner of the popular vote—no matter how small the margin—is not necessarily the winner of the electoral college vote.

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PUTTING IT TOGETHER: VOTING THEORY

There are four candidates for senior class president, Garcia, Lee, Nguyen, and Smith. Using a preference ballot, 75 ballots were cast, and the votes are shown below.

	20	3	8	16	28
1st choice	Garcia	Garcia	Lee	Nguyen	Smith
2nd choice	Lee	Nguyen	Nguyen	Garcia	Lee
3rd choice	Nguyen	Lee	Garcia	Lee	Garcia
4th choice	Smith	Smith	Smith	Smith	Nguyen

Now that the votes are in, it should be a simple matter to find out who won the election, right?

Well that depends on which voting system you choose.

Using plurality method, Smith wins. This is because Smith got 28 first place votes, while Garcia received $20 + 3 = 23$, Lee 8, and Nguyen 16. However, Smith was the very last choice for the majority of the students! This seems rather unfair, so let's explore another method.

The Borda count assigns points based on the ranking: 4 points for first place, 3 for second, 2 for third, and 1 for last.

	Garcia	Lee	Nguyen	Smith
1st choice (4 pts)	$23 \times 4 = 92$	$8 \times 4 = 32$	$16 \times 4 = 64$	$28 \times 4 = 112$
2nd choice (3 pts)	$16 \times 3 = 48$	$48 \times 3 = 144$	$11 \times 3 = 33$	$0 \times 3 = 0$
3rd choice (2 pts)	$36 \times 2 = 72$	$19 \times 2 = 38$	$20 \times 2 = 40$	$0 \times 2 = 0$
4th choice (1 pt)	$0 \times 1 = 0$	$0 \times 1 = 0$	$28 \times 1 = 28$	$47 \times 1 = 47$
Total Points	212	214	165	159

This time Smith comes in last and Lee is the winner. However the preference votes indicate that Lee is a lukewarm choice for most people. Only 8 students chose Lee as their first choice. Perhaps another voting method will reflect the students' preferences better.

Let's try instant runoff voting (IRV). This method proceeds in rounds, eliminating the candidate with the least number of first place votes at each round (with votes redistributed to voters' next choices) until a majority winner emerges. In the first round, Lee is immediately eliminated.

	$20 + 3 = 23$	$8 + 16 = 24$	28
1st choice	Garcia	Nguyen	Smith
2nd choice	Nguyen	Garcia	Garcia
3rd choice	Smith	Smith	Nguyen

There is still no majority winner. Garcia is eliminated next, which gives the election to Nguyen.

	$23 + 24 = 47$	28
1st choice	Nguyen	Smith
2nd choice	Smith	Nguyen

Finally, let's see if there is a Condorcet winner. We examine all one-on-one contests based on the original preference schedule. The table below summarizes the results. Each column shows the total number of ballots in which that candidate beats the candidate listed in each row. Remember, a majority of the 75 votes would be at least 38 (majority votes are highlighted in blue).

	Garcia	Lee	Nguyen	Smith
Garcia		36	24	28
Lee	39		19	28
Nguyen	51	56		28
Smith	47	47	47	

Garcia is the Condorcet winner with 39, 51, and 47 votes against Lee, Nguyen, and Smith, respectively.

Which voting method do you think is the most fair? The same voting preference schedule produced four different "winners." In a close election with many competing preferences, perhaps there is no clear winner. However a decision must be made.

This small example serves to show why understanding voting theory helps to put the election process in perspective. At the end of the day, one voting method must be selected and the winner decided according to those agreed-upon rules. Try out some other voting methods and see if you can make a case for who should be the senior class president!

GROWTH MODELS

WHY IT MATTERS: GROWTH MODELS

The following essay is adapted from blog posts on [emergent math](#) by Geoff Krall.

Can You Predict How Many Followers @charliesheen Has Right Now?

Sometime on March 1, 2011, Charlie Sheen joined twitter [at the suggestion](#) of Piers Morgan, who is apparently some type of person. By the time I was alerted of the existence of a [@charliesheen twitter feed](#), it was 4:04 PM Mountain Standard Time. Sheen had yet to tweet, but already had somehow amassed **over 100,000 followers**. He hadn't even put up an avatar of himself yet (and somehow the account was "verified").



What happened over the next several hours was nothing short of amazing. All you had to do was wait a few seconds or minutes and hit "refresh," and just watch the number of followers climb.

I did this for about 50 minutes and collected data along the way.

Time on March 1, 2011	# of Followers
4:04 PM	109,099
4:07 PM	112,497

4:09 PM	116,143
4:12 PM	120,195
4:14 PM	122,901
4:18 PM	127,643
4:21 PM	129,793
4:24 PM	133,357
4:29 PM	140,215
4:32 PM	144,103
4:38 PM	149,528
4:44 PM	153,848

Later that night, I refreshed every half-hour or so and watched the number climbing steadily. At this point, he had made his first tweet.



Time on March 1, 2011	# of Followers
6:21 PM	261,894
6:24 PM	275,638
6:49 PM	302,077
7:13 PM	324,595
7:38 PM	348,955
8:45 PM	429,904

Eventually I had to go to bed. So I did. And as soon as I woke up the next morning I went back to check Charlie Sheen's number of twitter followers.

Now I'd be checking every hour or so. Recorded here.

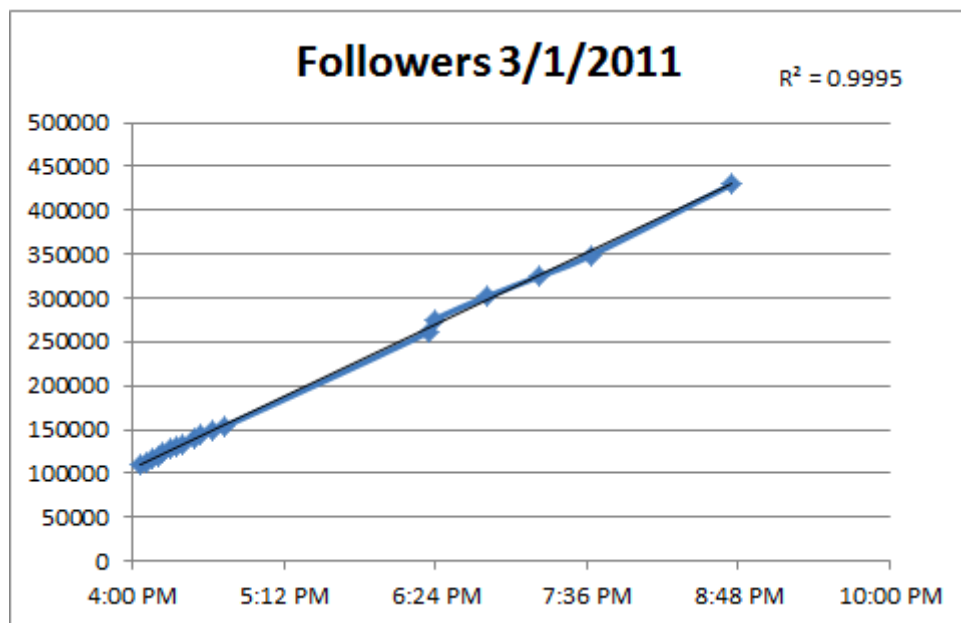
Time on March 2, 2011	# of Followers
6:45 AM	729,703
8:51 AM	799,008
9:36 AM	820,538
10:23 AM	841,940
11:16 AM	865,972
12:43 PM	902,291
1:47 PM	926,557
3:09 PM	957,259

The last datum I recorded was at 3:09 MST, at which point @charliesheen had **957,259 followers**. So in less than 24 hours, Charlie Sheen had **gained 800,000** followers on twitter.

I have two questions for you:

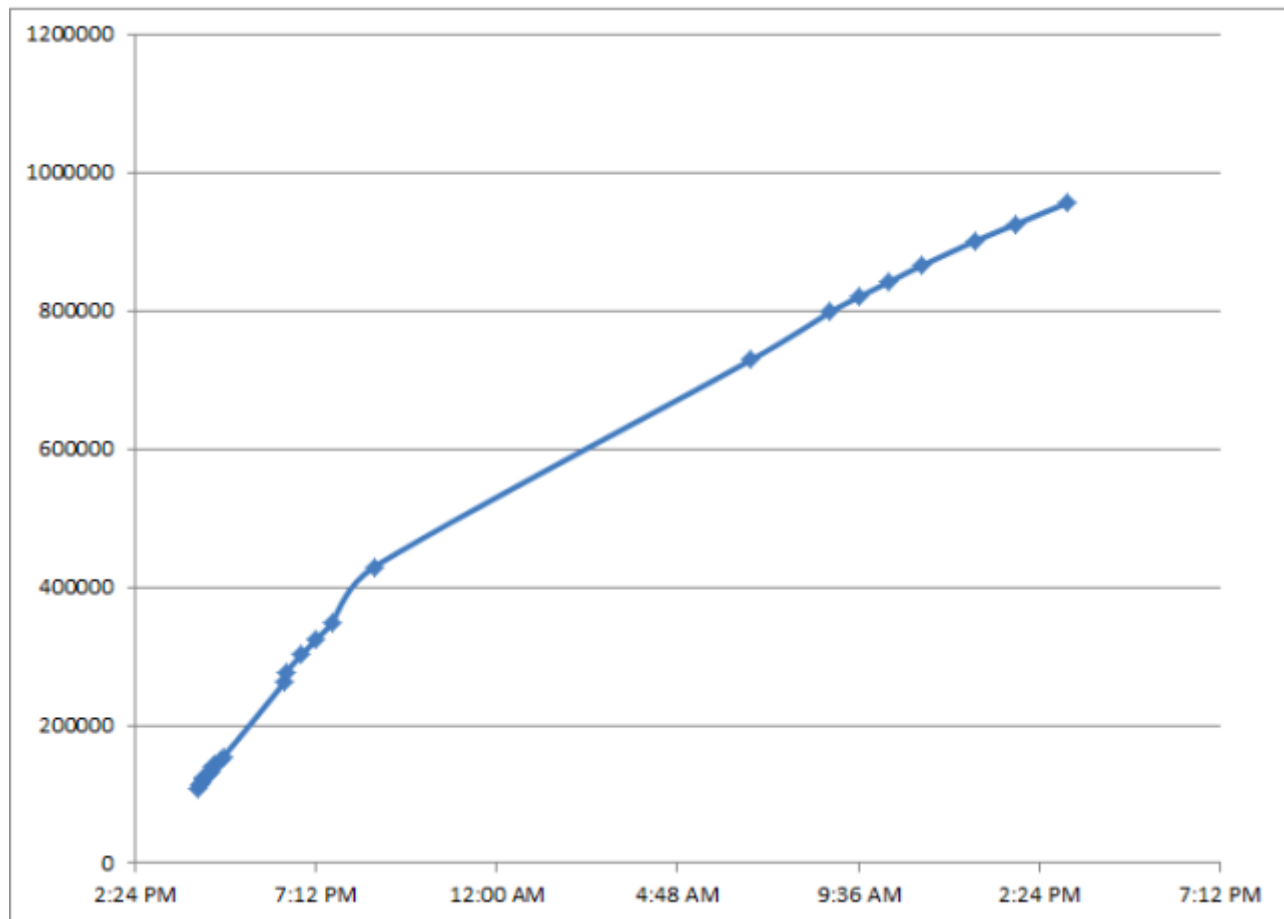
- **At what time March, 1, did Charlie Sheen join twitter? and,**
- **How many followers does he have right now?**

To answer, I'll break this up into a few different time pieces. Here's the plot of Charlie Sheen's followers as I have them, only for **3/1/2011**, the day he joined twitter.



Not only is that a linear fit, it's **very linear** (R-squared of 0.9995). I'm also shocked at how straight that line is. That is, there are not spikes of activity. I would have thought that in the evening, when everyone's sitting in front of their TVs with their iPad 2's there would be a marked increase in @charliesheen twitter followers.

Now let's look at the same graph, but with **all of Day 2** (3/2/2011) data included.



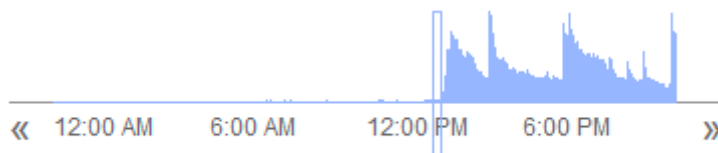
Unfortunately, I didn't check Sheen's twitter followers in the middle of the night, but it appears to have leveled off slightly going into Day 2 of the twitter account.

Now, I'm not 100% sure when the twitter feed account went live – in fact, that was one of the questions I asked last time – but I do have this:

Timeline

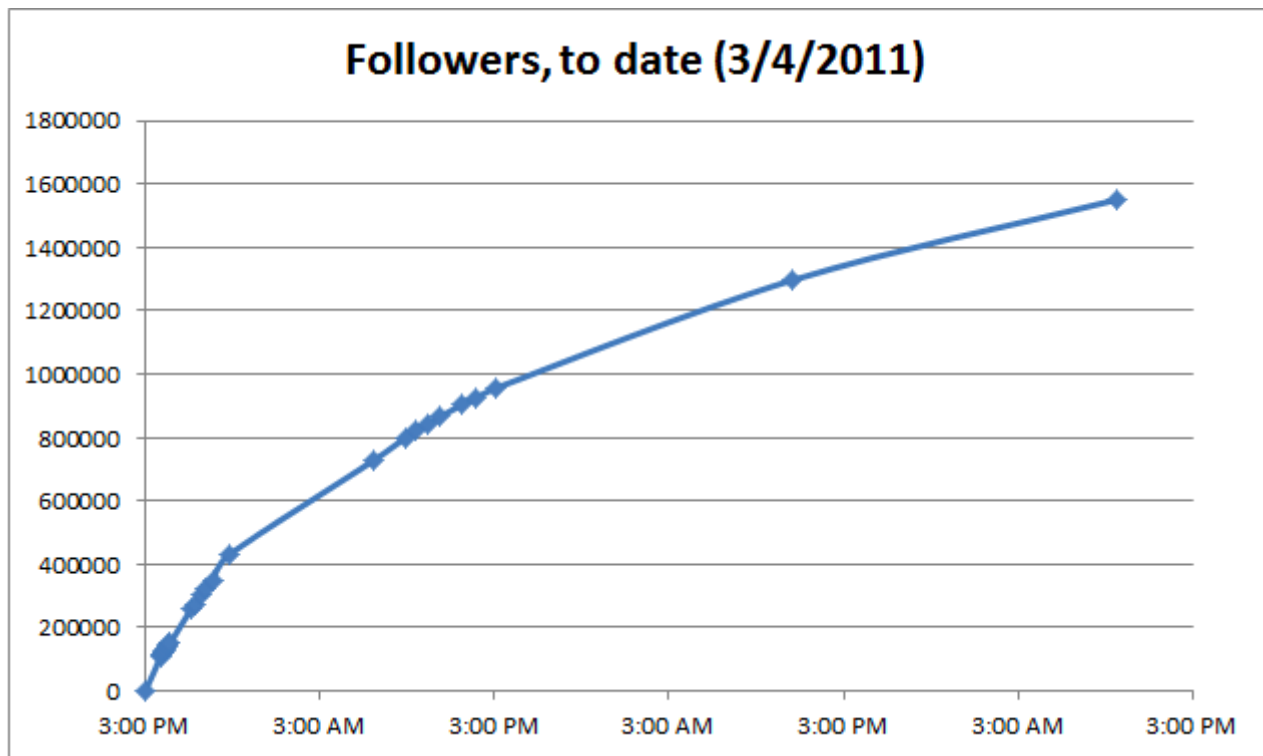
[2011](#) > [March](#) > 1 > 2:39 PM – 2:56 PM

[Now](#)



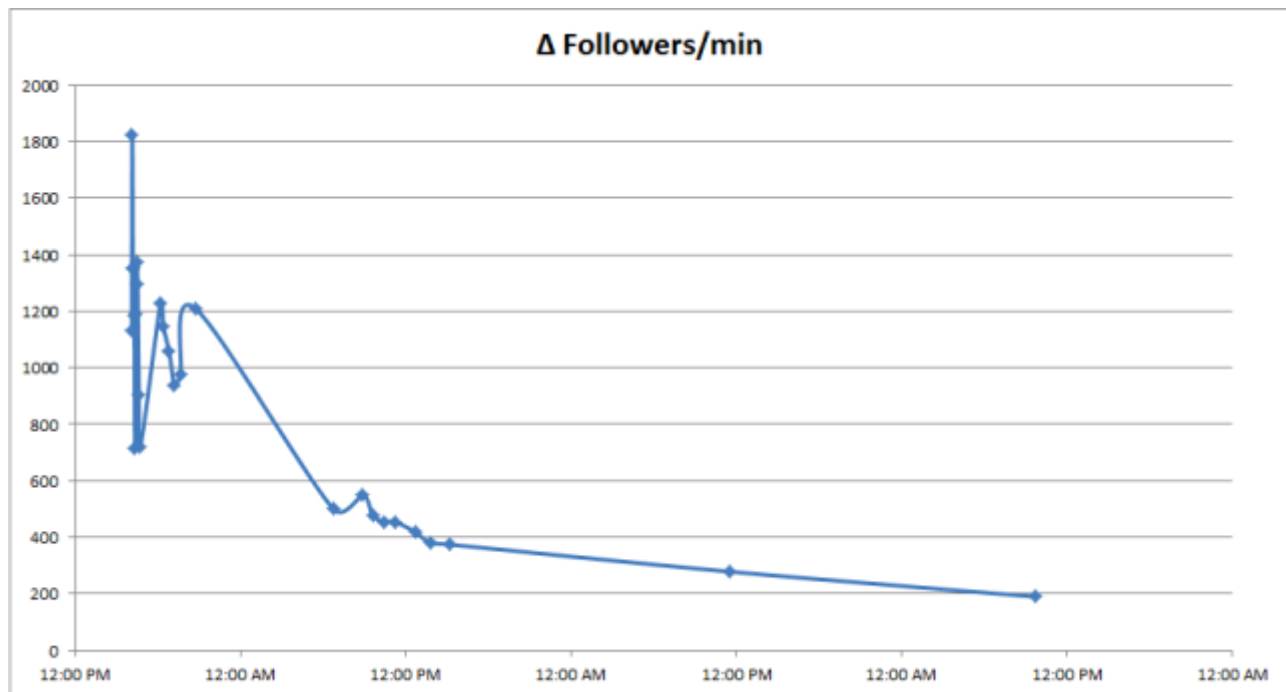
This is the Google realtime search feature. I would assume there's a strong correlation between the Google realtime searches for "@charliesheen" and the activation of the twitter account. So it looks like the account may have gone active sometime in the mid-afternoon of 3/1/2011. **Let's call it 3:00 PM.**

I took a couple of additional data points in the next couple of days, which leads me here:

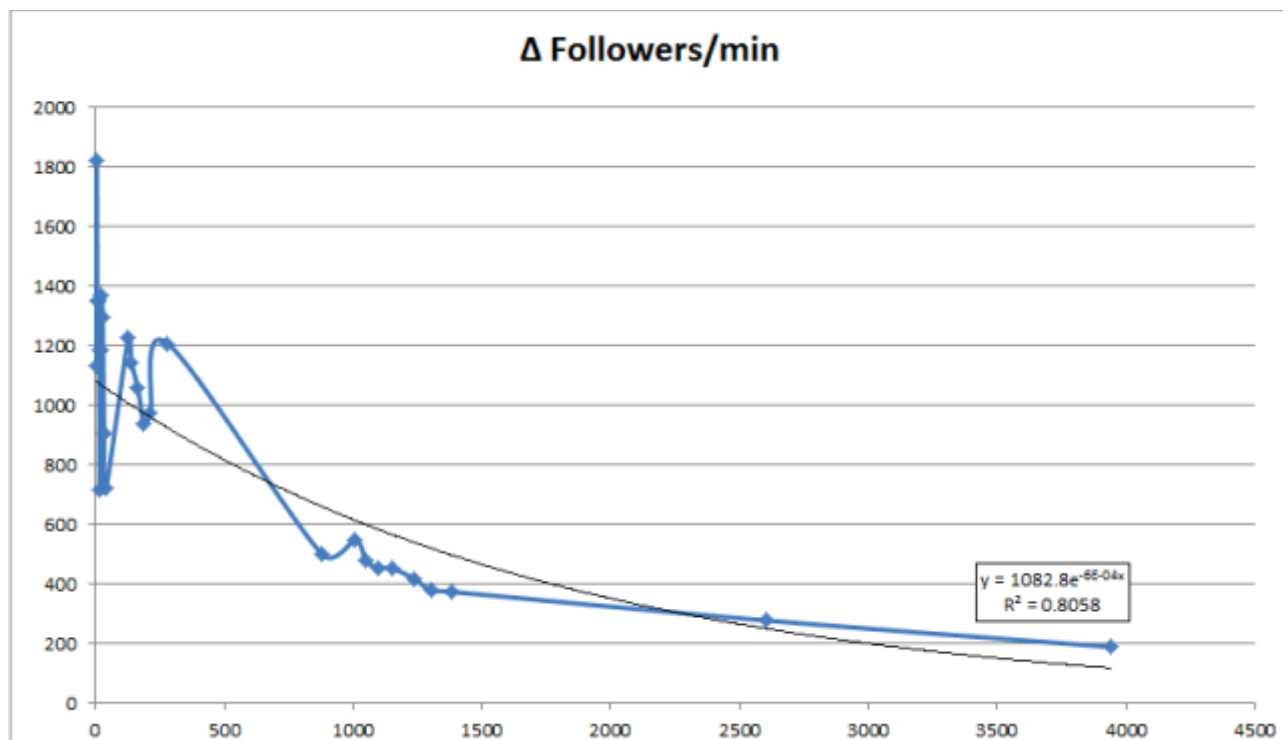


While still climbing steadily, **the curve is nowhere near as steep** as it was on Day 1, which is probably to be expected.

Hey, I bet the slope of the line over time would be interesting. And that would give us a clue as to where this thing is headed. Let's break it up into **(Change-in-followers per minute) vs. (time)**.



There's my exponential! We'll have to fudge the x-values a bit to get an actual equation and trendline.



And you know what? I bet if I tossed out a few of those early, “noisy” observation, we’d get a much better fitted curve. But still, it looks like you can pretty well describe the behavior of Charlie Sheen’s number of twitter followers with this **exponential decay curve**.

Now, what can we do with this? Was this just a ridiculous exercise by someone who spends too much time in front of his computer? Perhaps (probably). But are there any implications for the dramatic growth and tapering off of a (sort of crazy) celebrity’s twitter followers?

Remember in 2010, when **LeBron James** joined twitter before his announcement of where he was headed, dubbed “The Decision” on ESPN? His twitter followers had a similar explosion at the beginning before he even tweeted anything. I don’t have the data unfortunately. While everyone agrees that “The Decision” was a huge PR mistake, everyone also agrees that the ratings for “The Decision” were **through the roof**. Let’s put it this way: more people watched “The Decision” where James announced which team he would join than many actual NBA Finals games. **Did the simple creation of the twitter account boost the ratings?** What if “Two and a Half Men” came back on the air now? I have to assume that the ratings would be similarly boosted. **James Franco** did something similar before his hosting of the Oscars, joining twitter beforehand.

You think **agents** and TV and movie **execs** aren’t aware of the “twitter-effect?” Of course they are, and they’re probably also interested in some of the **hard data** that comes along with it.

Learning Objectives

Linear and Geometric Growth

- Build a recursive equation that models linear or exponential growth
- Build an explicit equation that models linear or exponential growth
- Make predictions using linear and exponential growth models

Logarithms and Logistic Growth

- Use logarithms to solve exponential growth models for time
- Identify the carrying capacity and growth rate of the logistic growth model
- Use the logistic growth model to make predictions

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- Can you predict how many followers @charliesheen has right now? ; Exponential growth? (Part 1). **Authored by:** Geoff Krall. **Provided by:** emergent math. **Located at:** <https://emergentmath.com/2011/03/03/can-you-predict-how-many-followers-charliesheen-has-right-now-exponential-growth-part-1/>. **License:** [CC BY-NC: Attribution-NonCommercial](#)
- Can you predict how many followers @charliesheen has right now? ; Exponential growth? (Part 2). **Authored by:** Geoff Krall. **Provided by:** emergent math. **Located at:** <https://emergentmath.com/2011/03/04/can-you-predict-how-many-followers-charliesheen-has-right-now-exponential-growth-part-2/>. **License:** [CC BY-NC: Attribution-NonCommercial](#)
- Learning objectives: Growth Models. **Authored by:** David Lippman. **Located at:** <http://www.opentextbookstore.com/mathinsociety/>. **Project:** Math in Society. **License:** [CC BY-SA: Attribution-ShareAlike](#)

INTRODUCTION: LINEAR AND GEOMETRIC GROWTH

Learning Objectives

- Determine whether data or a scenario describe linear or geometric growth
- Identify growth rates, initial values, or point values expressed verbally, graphically, or numerically, and translate them into a format usable in calculation
- Calculate recursive and explicit equations for linear and geometric growth given sufficient information, and use those equations to make predictions

Constant change is the defining characteristic of linear growth. Plotting coordinate pairs associated with constant change will result in a straight line, the shape of linear growth. In this section, we will formalize a way to describe linear growth using mathematical terms and concepts. By the end of this section, you will be able to write both a recursive and explicit equations for linear growth given starting conditions, or a constant of change.

You will also be able to recognize the difference between linear and geometric growth given a graph or an equation.



LINEAR (ALGEBRAIC) GROWTH

Predicting Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?



While you could probably solve both of these questions without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$P_0 = 437$$

$$P_n = P_{n-1} + 32$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1} . Using this relationship, we could calculate:

$$P_1 = P_0 + 32 = 437 + 32 = 469$$

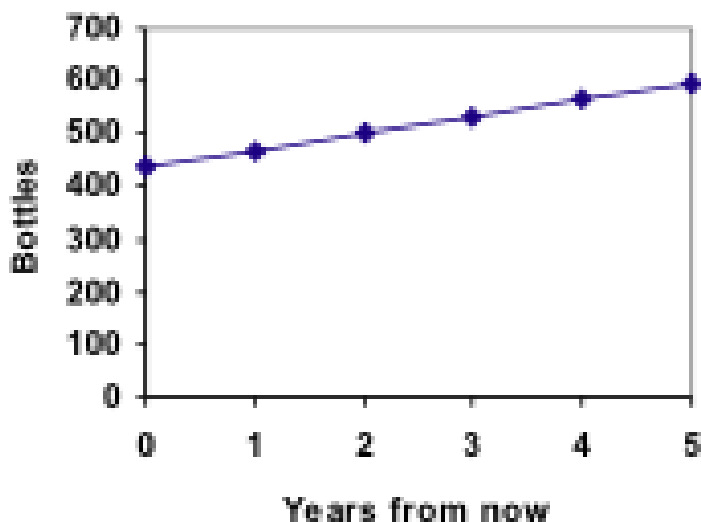
$$P_2 = P_1 + 32 = 469 + 32 = 501$$

$$P_3 = P_2 + 32 = 501 + 32 = 533$$

$$P_4 = P_3 + 32 = 533 + 32 = 565$$

$$P_5 = P_4 + 32 = 565 + 32 = 597$$

We have answered the question of how many bottles Marco will have in 5 years.



However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate P_n directly, without needing to know P_{n-1} . While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$P_1 = 437 + 32 = 437 + 1(32)$$

$$P_2 = P_1 + 32 = 437 + 32 + 32 = 437 + 2(32)$$

$$P_3 = P_2 + 32 = (437 + 2(32)) + 32 = 437 + 3(32)$$

$$P_4 = P_3 + 32 = (437 + 3(32)) + 32 = 437 + 4(32)$$

You can probably see the pattern now, and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for P_n and solving for n

$$1000 = 437 + 32n$$

$$563 = 32n$$

$$n = 563/32 = 17.59$$

So Marco will reach 1000 bottles in 18 years.

The steps of determining the formula and solving the problem of Marco's bottle collection are explained in detail in the following videos.

Watch this video online: https://youtu.be/SJcAjN-HL_I

Watch this video online: https://youtu.be/4Two_oduhrA

Watch this video online: <https://youtu.be/pZ4u3j8Vmzo>

In this example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

Linear Growth

If a quantity starts at size P_0 and grows by d every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form

$$P_n = P_{n-1} + d$$

Explicit form

$$P_n = P_0 + d n$$

In this equation, d represents the **common difference** – the amount that the population changes each time n increases by 1.

Connection to Prior Learning: Slope and Intercept

You may recognize the common difference, d , in our linear equation as *slope*. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, b was the y -intercept, or the y value when x was zero. In the form of the equation we're using, we are using P_0 to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as "rise over run," or the change in y divided by the change in x . Either way, it represents the same thing as the common difference, d , we are using – the amount the output P_n changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + d n$ mean the same thing and can be used the same ways. We're just writing it somewhat differently.

Examples

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

Answer

To begin, we need to define how we're going to measure n . Remember that P_0 is the population when $n = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003.

Then $P_0 = 12,000$.

Next we need to find d . Remember d is the growth per time period, in this case growth per year. Between the two measurements, the population grew by $15,000 - 12,000 = 3,000$, but it took $2007 - 2003 = 4$ years to grow that much. To find the growth per year, we can divide: $3000 \text{ elk} / 4 \text{ years} = 750 \text{ elk in 1 year}$.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{15,000 - 12,000}{2007 - 2003} = \frac{3000}{4} = 750$$

We can now write our equation in whichever form is preferred.

Recursive form

$$P_0 = 12,000$$

$$P_n = P_{n-1} + 750$$

Explicit form

$$P_n = 12,000 + 750n$$

To answer the question, we need to first note that the year 2014 will be $n = 11$, since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

$$P_{11} = 12,000 + 750(11) = 20,250 \text{ elk}$$

View more about this example here.

Watch this video online: <https://youtu.be/J1XqqIKzYGs>

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below. (Note:

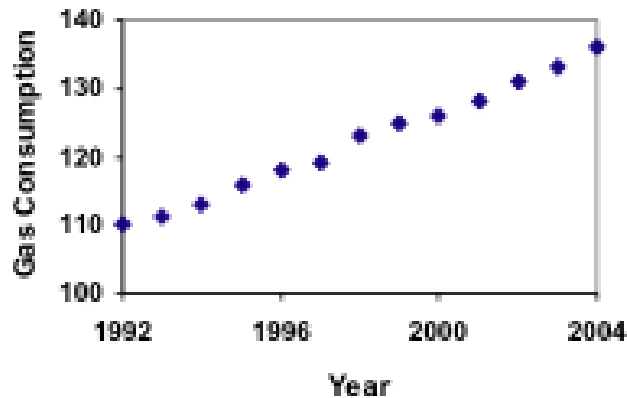
http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html) Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?

Year	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
------	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----

Consumption (billion of gallons)	110	111	113	116	118	119	123	125	126	128	131	133	136
--	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----

Answer

Plotting this data, it appears to have an approximately linear relationship:



While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

Letting $n = 0$ correspond with 1993 would give $P_0 = 111$ billion gallons.

To find d , we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of $133 - 111 = 22$ billion gallons, over 10 years. This gives us an average change of $22 \text{ billion gallons} / 10 \text{ year} = 2.2$ billion gallons per year.

Equivalently,

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133 - 111}{10 - 0} = \frac{22}{10} = 2.2 \text{ billion gallons per year}$$

We can now write our equation in whichever form is preferred.

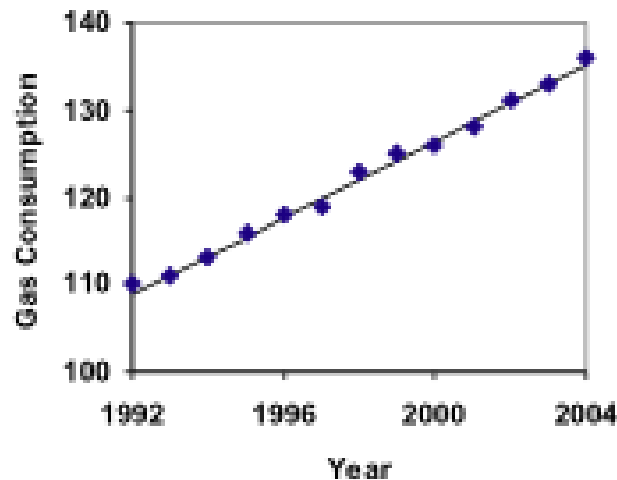
Recursive form

$$P_0 = 111$$

$$P_n = P_{n-1} + 2.2$$

Explicit form

$$P_n = 111 + 2.2n$$



Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

$n = 23$ (2016 – 1993 = 23 years later)

$$P_{23} = 111 + 2.2(23) = 161.6$$

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set $P_n = 200$, and solve for n :

$$P_n = 200$$

Replace P_n with our model

$$111 + 2.2n = 200$$

Subtract 111 from both sides

$$2.2n = 89$$

Divide both sides by 2.2

$$n = 40.4545$$

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

The steps for reaching this answer are detailed in the following video.

Watch this video online: <https://youtu.be/ApFxDWd6lbE>

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $P_n = 70 + 30n$. What does this equation tell us?

Answer

The value for P_0 in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for d in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month.

The explanation for this example is detailed below.

Watch this video online: <https://youtu.be/0Uwz5dmlTtk>

Try It Now

The number of stay-at-home fathers in Canada has been growing steadily (Note: <http://www.fira.ca/article.php?id=140>). While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number in 2020.

Year	1976	1984	1991	2000	2010
# of Stay -at-home fathers	20610	28725	43530	47665	53555

Answer

Letting $n = 0$ correspond with 1976, then $P_0 = 20,610$.

From 1976 to 2010 the number of stay-at-home fathers increased by $53,555 - 20,610 = 32,945$

This happened over 34 years, giving a common different d of $32,945 / 34 = 969$.

$$P_n = 20,610 + 969n$$

Predicting for 2020, we use $n = 44$, $P(44) = 20,610 + 969(44) = 63,246$ stay-at-home fathers in 2020.

Visit this page in your course online to practice before taking the quiz.

When Good Models Go Bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example

Suppose a four year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year.

We can set up a growth model, with $n = 0$ corresponding to 4 years old.

Recursive form

$$P_0 = 39$$

$$P_n = P_{n-1} + 2.5$$

Explicit form

$$P_n = 39 + 2.5n$$

So at 6 years old, we would expect him to be

$$P_2 = 39 + 2.5(2) = 44 \text{ inches tall}$$

Any mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$$P_{46} = 39 + 2.5(46) = 154 \text{ inches tall} = 12.8 \text{ feet tall!}$$

When using any mathematical model, we have to consider which inputs are reasonable to use. Whenever we **extrapolate**, or make predictions into the future, we are assuming the model will continue to be valid.

View a video explanation of this breakdown of the linear growth model here.

Watch this video online: <https://youtu.be/6zfXCsmcDzl>

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EXPONENTIAL (GEOMETRIC) GROWTH

Population Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. If there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year.



Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1 ? The new population will be the old population, plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as “the original 100% plus an additional 10%.”

For our fish population,

$$P_1 = 1.10(1000) = 1100$$

We could then calculate the population in later years:

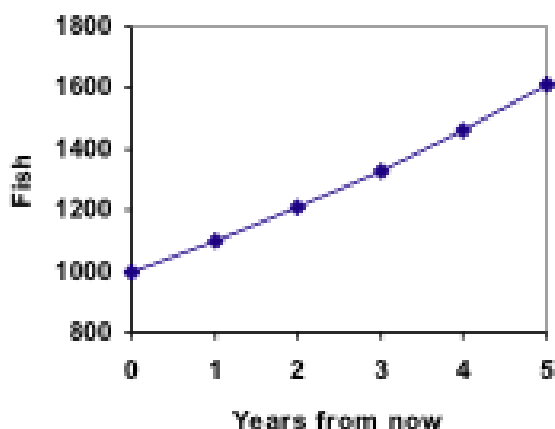
$$P_2 = 1.10P_1 = 1.10(1100) = 1210$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331$$

Notice that in the first year, the population grew by 100 fish; in the second year, the population grew by 110 fish; and in the third year the population grew by 121 fish.

While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

Graphing these values we see that this growth doesn't quite appear linear.



A walkthrough of this fish scenario can be viewed here:

Watch this video online: <https://youtu.be/3BiU7lhvxvg>

To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$P_1 = 1.10P_0 = 1.10(1000)$$

$$P_2 = 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000)$$

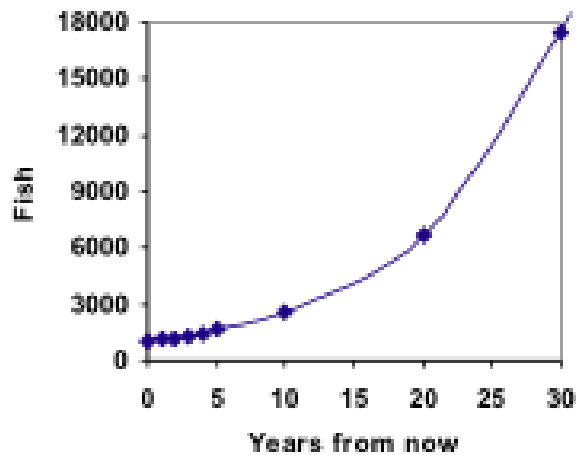
$$P_3 = 1.10P_2 = 1.10(1.10^2(1000)) = 1.10^3(1000)$$

$$P_4 = 1.10P_3 = 1.10(1.10^3(1000)) = 1.10^4(1000)$$

Observing a pattern, we can generalize the explicit form to be:

$$P_n = 1.10^n(1000), \text{ or equivalently, } P_n = 1000(1.10^n)$$

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:



$$P_{10} = 1.10_{10}(1000) = 2594$$

$$P_{20} = 1.10_{20}(1000) = 6727$$

$$P_{30} = 1.10_{30}(1000) = 17449$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years, compared to almost 18,000 with this percent-based growth, called **exponential growth**.

A video demonstrating the explicit model of this fish story can be viewed here:

Watch this video online: <https://youtu.be/tg2ysaZ8agY>

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Exponential Growth

If a quantity starts at size P_0 and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form

$$P_n = (1+r) P_{n-1}$$

Explicit form

$$P_n = (1+r)^n P_0 \quad \text{or equivalently, } P_n = P_0 (1+r)^n$$

We call r the **growth rate**.

The term $(1+r)$ is called the **growth multiplier**, or common ratio.

Example

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

Answer

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$$P_6 = (1+0.03)^6 (245,000) = 1.19405(245,000) = 292,542.25$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

The following video explains this example in detail.

Watch this video online: <https://youtu.be/CDI4xS65rxY>

Evaluating exponents on the calculator

To evaluate expressions like $(1.03)^6$, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically either labeled like:

\wedge , y^x , or x^y .

To evaluate 1.03^6 we'd type $1.03 \wedge 6$, or $1.03 y^x 6$. Try it out – you should get an answer around 1.1940523.

Try It Now

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Answer

Using $n = 0$ corresponding with 2008, $P_{12} = (1 + 0.0134)^{12}(1.14) =$ about 1.337 billion people in 2020

Visit this page in your course online to practice before taking the quiz.

Examples

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

Answer

In the equation, $P_0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was \$4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was \$4,600, and is expected to grow by 7.2% each year.

View the following to see this example worked out.

Watch this video online: <https://youtu.be/T8Yz94De5UM>

In 1990, residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons (Note: <http://www.eia.doe.gov/oiaf/1605/ggrpt/carbon.html>). If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Answer

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P_0 = 962$ (million metric tons of CO_2). In this problem, we are not given the growth rate, but instead are given that $P_{10} = 1182$.

When $n = 10$, the explicit equation looks like:

$$P_{10} = (1+r)^{10} P_0$$

We know the value for P_0 , so we can put that into the equation:

$$P_{10} = (1+r)^{10} 962$$

We also know that $P_{10} = 1182$, so substituting that in, we get

$$1182 = (1+r)^{10} 962$$

We can now solve this equation for the growth rate, r . Start by dividing by 962.

$$\frac{1182}{962} = (1+r)^{10} \quad \text{Take the 10th root of both sides}$$

$$\sqrt[10]{\frac{1182}{962}} = 1+r \quad \text{Subtract 1 from both sides}$$

$$r = \sqrt[10]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding P_{60}

$$P_{60} = (1+0.0208)^{60} 962 = 3308.4 \text{ million metric tons of } \text{CO}_2 \text{ in 2050}$$

View more about this example here.

Watch this video online: <https://youtu.be/9Zu2uONfLkQ>

Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

If you need to round, **keep at least three significant digits** – numbers after any leading zeros. So 0.4162 could be reasonably rounded to 0.416. A growth rate of 0.001027 could be reasonably rounded to 0.00103.

Evaluating roots on the calculator

In the previous example, we had to calculate the 10th root of a number. This is different than taking the basic square root, $\sqrt{}$. Many scientific calculators have a button for general roots. It is typically labeled like:

$$\sqrt[n]{x}$$

To evaluate the 3rd root of 8, for example, we'd either type $3 \sqrt[3]{8}$, or $8 \sqrt[3]{3}$, depending on the calculator.

Try it on yours to see which to use – you should get an answer of 2.

If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that:

$$\sqrt[n]{a} = a^{\frac{1}{n}}.$$

So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate $8^{1/3}$. To do this, type:

$$8 \text{ y}^x (1 \div 3)$$

The parentheses tell the calculator to divide 1/3 before doing the exponent.

Try It Now

The number of users on a social networking site was 45 thousand in February when they officially went public, and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should they expect two years after they went public?

Answer

Here we will measure n in months rather than years, with $n = 0$ corresponding to the February when they went public. This gives $P_0 = 45$ thousand. October is 8 months later, so $P_8 = 60$.

$$P_8 = (1 + r)^8 P_0$$

$$60 = (1 + r)^8 45$$

$$\frac{60}{45} = (1 + r)^8$$

$$\sqrt[8]{\frac{60}{45}} = 1 + r$$

$$r = \sqrt[8]{\frac{60}{45}} - 1 = 0.0366 \text{ or } 3.66$$

The general explicit equation is $P_n = (1.0366)^n 45$. Predicting 24 months after they went public gives

$$P_{24} = (1.0366)^{24} 45 = 106.63 \text{ thousand users.}$$

Visit this page in your course online to practice before taking the quiz.

Example

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Answer

Again we will get $n = 0$ correspond with 1990, giving $P_0 = 962$. To find d , we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$$P_{10} = P_0 + 10d$$

We know the value for P_0 , so we can put that into the equation:

$$P_{10} = 962 + 10d$$

Since we know that $P_{10} = 1182$, substituting that in we get

$$1182 = 962 + 10d$$

We can now solve this equation for the common difference, d .

$$1182 - 962 = 10d$$

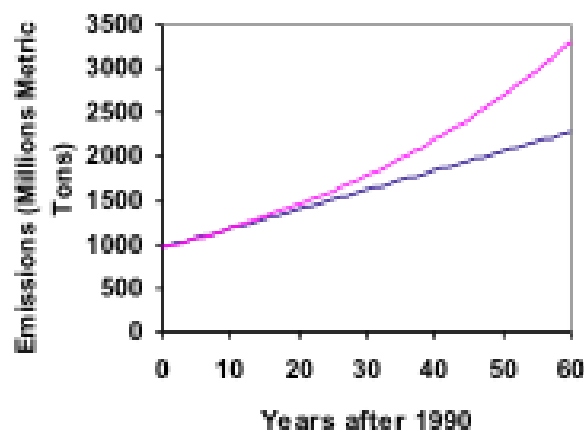
$$220 = 10d$$

$$d = 22$$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$$P_{60} = 962 + 22(60) = 2282 \text{ million metric tons.}$$

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.



A demonstration of this example can be seen in the following video.

Watch this video online: <https://youtu.be/yiuZoiRMtYM>

So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

1. Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
2. Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

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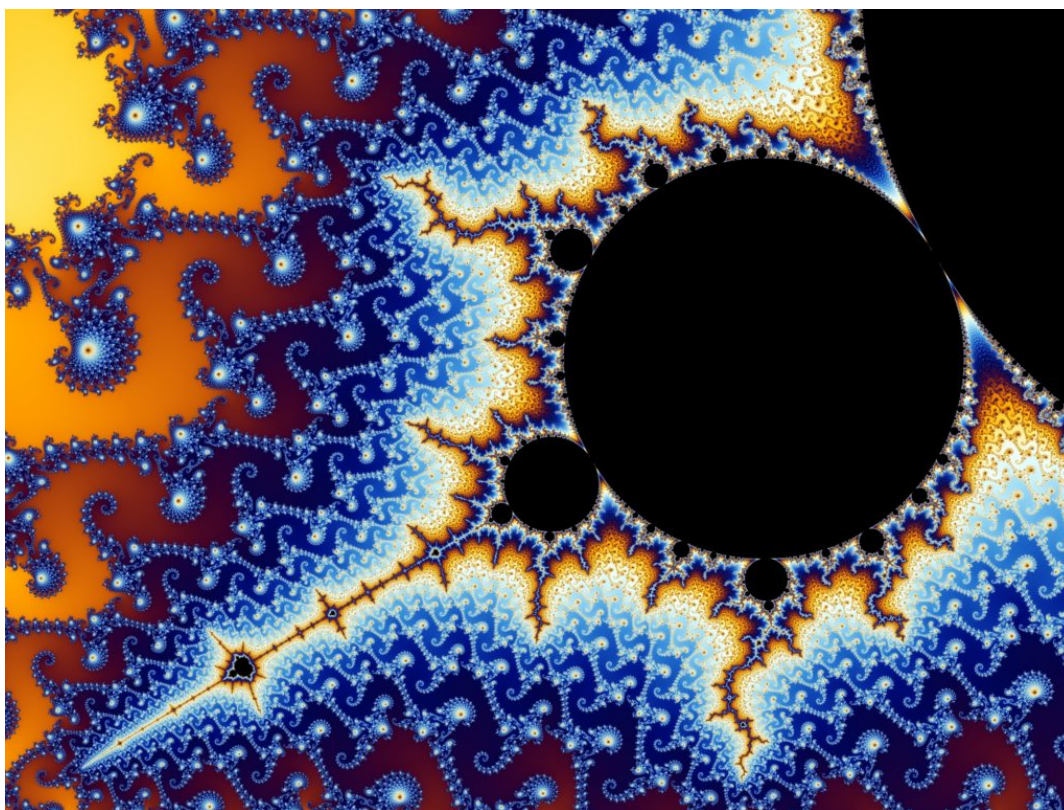
INTRODUCTION: LOGARITHMS AND LOGISTIC GROWTH

Learning Objectives

- Evaluate and rewrite logarithms using the properties of logarithms
- Use the properties of logarithms to solve exponential models for time
- Identify the carrying capacity in a logistic growth model
- Use a logistic growth model to predict growth

In a confined environment the growth rate of a population may not remain constant. In a lake, for example, there is some *maximum sustainable population* of fish, also called a **carrying capacity**. In this section, we will develop a model that contains a carrying capacity term, and use it to predict growth under constraints.

Because resources are typically limited in systems, these types of models are much more common than linear or geometric growth.



The famous Mandelbrot set, a fractal whose growth is constrained.

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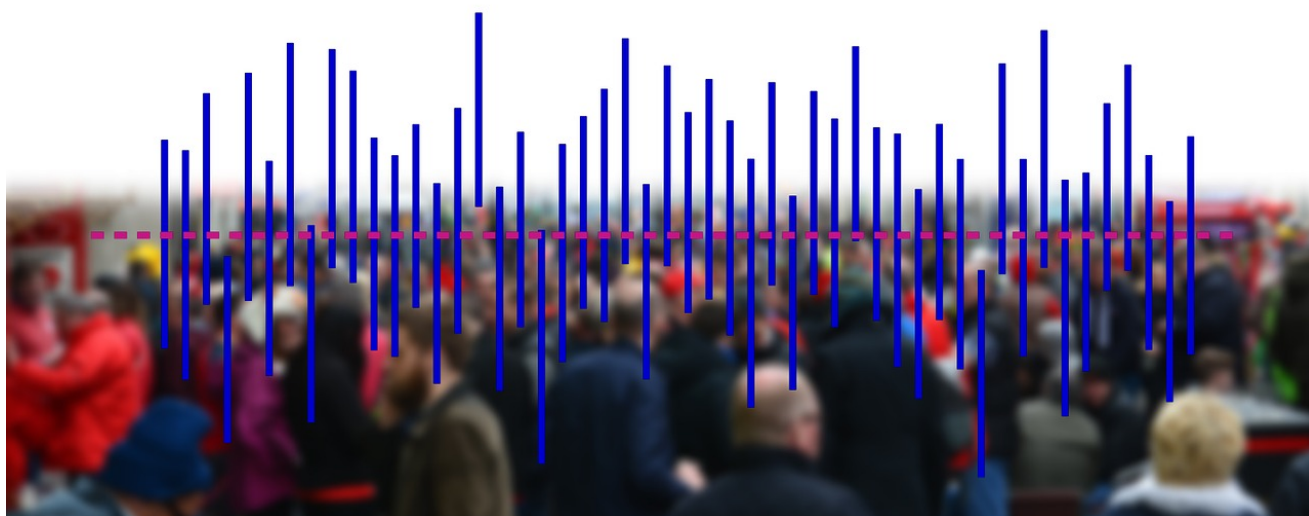
SOLVE EXPONENTIALS FOR TIME: LOGARITHMS

Reversing an Exponent

Earlier, we found that since Olympia, WA had a population of 245 thousand in 2008 and had been growing at 3% per year, the population could be modeled by the equation

$$P_n = (1+0.03)^n (245,000), \text{ or equivalently, } P_n = 245,000(1.03)^n.$$

Using this equation, we were able to predict the population in the future.



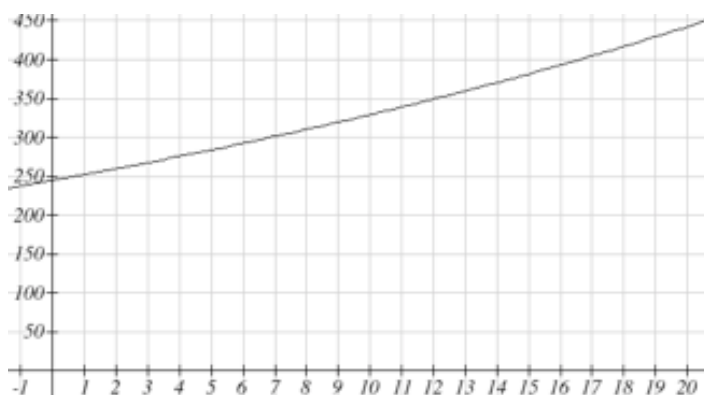
Suppose we wanted to know when the population of Olympia would reach 400 thousand. Since we are looking for the year n when the population will be 400 thousand, we would need to solve the equation

$$400,000 = 245,000(1.03)^n$$

Dividing both sides by 245,000 gives

$$1.6327 = 1.03^n$$

One approach to this problem would be to create a table of values, or to use technology to draw a graph to estimate the solution.



From the graph, we can estimate that the solution will be around 16 to 17 years after 2008 (2024 to 2025). This is pretty good, but we'd really like to have an algebraic tool to answer this question. To do that, we need to introduce a new function that will undo exponentials, similar to how a square root undoes a square. For exponentials, the function we need is called a logarithm. It is the inverse of the exponential, meaning it undoes the exponential. While there is a whole family of logarithms with different bases, we will focus on the common log, which is based on the exponential 10^x .

Common Logarithm

The common logarithm, written $\log(x)$, undoes the exponential 10^x

This means that $\log(10^x) = x$, and likewise $10^{\log(x)} = x$

This also means the statement $10^a = b$ is equivalent to the statement $\log(b) = a$

$\log(x)$ is read as “log of x ”, and means “the logarithm of the value x ”. It is important to note that this is *not* multiplication – the log doesn’t mean anything by itself, just like $\sqrt{}$ doesn’t mean anything by itself; it has to be applied to a number.

Example

Evaluate each of the following

- $\log(100)$
- $\log(1000)$
- $\log(10000)$
- $\log(1/100)$
- $\log(1)$

Answer

- $\log(100)$ can be written as $\log(10^2)$. Since the log undoes the exponential, $\log(10^2) = 2$
- $\log(1000) = \log(10^3) = 3$
- $\log(10000) = \log(10^4) = 4$
- Recall that $x^{-n} = \frac{1}{x^n}$. $\log\left(\frac{1}{100}\right) = \log(10^{-2}) = -2$
- Recall that $x^0 = 1$. $\log(1) = \log(10^0) = 0$

Visit this page in your course online to practice before taking the quiz.

It is helpful to note that from the first three parts of the previous example that the number we're taking the log of has to get *10 times bigger* for the log to increase in value by 1.

Of course, most numbers cannot be written as a nice simple power of 10. For those numbers, we can evaluate the log using a scientific calculator with a log button.

Example

Evaluate $\log(300)$

Answer

Using a calculator, $\log(300)$ is approximately 2.477121

With an equation, just like we can add a number to both sides, multiply both sides by a number, or square both sides, we can also take the logarithm of both sides of the equation and end up with an equivalent equation. This will allow us to solve some simple equations.

Examples

1. Solve $10^x = 1000$
2. Solve $10^x = 3$
3. Solve $2(10^x) = 8$

Answer

1. Taking the log of both sides gives $\log(10^x) = \log(1000)$
 - Since the log undoes the exponential, $\log(10^x) = x$. Similarly $\log(1000) = \log(10^3) = 3$.
 - The equation simplifies then to $x = 3$.
2. Taking the log of both sides gives $\log(10^x) = \log(3)$.
 - On the left side, $\log(10^x) = x$, so $x = \log(3)$.
 - We can approximate this value with a calculator. $x \approx 0.477$
3. Here we would first want to isolate the exponential by dividing both sides of the equation by 2, giving $10^x = 4$.
 - Now we can take the log of both sides, giving $\log(10^x) = \log(4)$, which simplifies to $x = \log(4) \approx 0.602$

This approach allows us to solve exponential equations with powers of 10, but what about problems like $2 = 1.03^n$ from earlier, which have a base of 1.03? For that, we need the exponent property for logs.

Properties of Logs: Exponent Property

$$\log(A^r) = r \log(A)$$

To show why this is true, we offer a proof.

Since the logarithm and exponential undo each other, $10^{\log A} = A$.

$$\text{So } A^r = (10^{\log A})^r$$

Utilizing the exponential rule that states $(x^a)^b = x^{ab}$,

$$A^r = (10^{\log A})^r = 10^{r \log A}$$

$$\text{So then } \log(A^r) = \log(10^{r \log A})$$

Again utilizing the property that the log undoes the exponential on the right side yields the result

$$\log(A^r) = r \log A$$

Example

Rewrite $\log(25)$ using the exponent property for logs.

Answer

$$\log(25) = \log(5^2) = 2\log(5)$$

Visit this page in your course online to practice before taking the quiz.

This property will finally allow us to answer our original question.

Solving exponential equations with logarithms

1. Isolate the exponential. In other words, get it by itself on one side of the equation. This usually involves dividing by a number multiplying it.
2. Take the log of both sides of the equation.
3. Use the exponent property of logs to rewrite the exponential with the variable exponent multiplying the logarithm.
4. Divide as needed to solve for the variable.

Example

If Olympia is growing according to the equation, $P_n = 245(1.03)^n$, where n is years after 2008, and the population is measured in thousands. Find when the population will be 400 thousand.

Answer

We need to solve the equation

$$400 = 245(1.03)^n$$

exponential

$$1.633 = 1.03^n$$

$$\log(1.633) = \log(1.03^n)$$

$$\log(1.633) = n \log(1.03)$$

$$\frac{\log(1.633)}{\log(1.03)} = n$$

$$n \approx 16.591$$

Begin by dividing both sides by 245 to isolate the

Now take the log of both sides

Use the exponent property of logs on the right side

Now we can divide by $\log(1.03)$

We can approximate this value on a calculator

A full walkthrough of this problem is available here.

Watch this video online: <https://youtu.be/liNffAACIU8>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Alternatively, after applying the exponent property of logs on the right side, we could have evaluated the logarithms to decimal approximations and completed our calculations using those approximations, as you'll see in the next example. While the final answer may come out slightly differently, as long as we keep enough significant values during calculation, our answer will be close enough for most purposes.

Example

Polluted water is passed through a series of filters. Each filter removes 90% of the remaining impurities from the water. If you have 10 million particles of pollutant per gallon originally, how many filters would the water need to be passed through to reduce the pollutant to 500 particles per gallon?

Answer

In this problem, our "population" is the number of particles of pollutant per gallon. The initial pollutant is 10 million particles per gallon, so $P_0 = 10,000,000$. Instead of changing with time, the pollutant changes with the number of filters, so n will represent the number of filters the water passes through.

Also, since the amount of pollutant is *decreasing* with each filter instead of increasing, our "growth" rate will be negative, indicating that the population is decreasing instead of increasing, so $r = -0.90$.

We can then write the explicit equation for the pollutant:

$$P_n = 10,000,000(1 - 0.90)^n = 10,000,000(0.10)^n$$

To solve the question of how many filters are needed to lower the pollutant to 500 particles per gallon, we can set P_n equal to 500, and solve for n .

$$500 = 10,000,000(0.10)^n$$

$$0.00005 = 0.10^n$$

$$\log(0.00005) = \log(0.10^n)$$

$$\log(0.00005) = n \log(0.10)$$

$$-4.301 = n(-1)$$

$$4.301 = n$$

Divide both sides by 10,000,000

Take the log of both sides

Use the exponent property of logs on the

Evaluate the logarithms to a decimal

Divide by -1, the value multiplying n

It would take about 4.301 filters. Of course, since we probably can't install 0.3 filters, we would need to use 5 filters to bring the pollutant below the desired level.

More details about solving this scenario are available in this video.

Watch this video online: <https://youtu.be/sLLu0u1YgM0>

Try It Now

India had a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, when will India's population reach 1.2 billion?
Visit this page in your course online to practice before taking the quiz.

TIP

When you are solving growth problems, use the language in the question to determine whether you are solving for time, future value, present value or growth rate. Questions that uses words like “when”, “what year”, or “how long” are asking you to solve for time and you will need to use logarithms to solve them because the time variable in growth problems is in the exponent.

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LOGISTIC GROWTH

Limits on Exponential Growth

In our basic exponential growth scenario, we had a recursive equation of the form

$$P_n = P_{n-1} + r P_{n-1}$$

In a confined environment, however, the growth rate may not remain constant. In a lake, for example, there is some *maximum sustainable population* of fish, also called a **carrying capacity**.

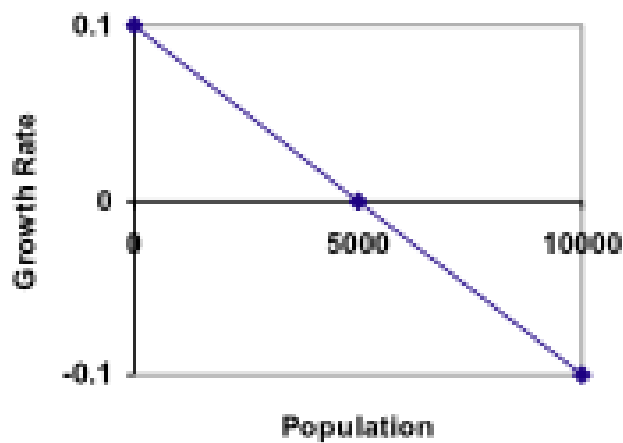
Carrying Capacity

The **carrying capacity**, or **maximum sustainable population**, is the largest population that an environment can support.



For our fish, the carrying capacity is the largest population that the resources in the lake can sustain. If the population in the lake is far below the carrying capacity, then we would expect the population to grow essentially exponentially. However, as the population approaches the carrying capacity, there will be a scarcity of food and space available, and the growth rate will decrease. If the population exceeds the carrying capacity, there won't be enough resources to sustain all the fish and there will be a negative growth rate, causing the population to decrease back to the carrying capacity.

If the carrying capacity was 5000, the growth rate might vary something like that in the graph shown.



Note that this is a linear equation with intercept at 0.1 and slope $-\frac{0.1}{5000}$, so we could write an equation for this adjusted growth rate as:

$$r_{adjusted} = 0.1 - \frac{0.1}{5000}P = 0.1 \left(1 - \frac{P}{5000}\right)$$

Substituting this in to our original exponential growth model for r gives

$$P_n = P_{n-1} + 0.1 \left(1 - \frac{P_{n-1}}{5000}\right) P_{n-1}$$

View the following for a detailed explanation of the concept.

Watch this video online: https://youtu.be/-6VLXCTkP_c

Logistic Growth

If a population is growing in a constrained environment with carrying capacity K , and absent constraint would grow exponentially with growth rate r , then the population behavior can be described by the logistic growth model:

$$P_n = P_{n-1} + r \left(1 - \frac{P_{n-1}}{K}\right) P_{n-1}$$

Unlike linear and exponential growth, logistic growth behaves differently if the populations grow steadily throughout the year or if they have one breeding time per year. The recursive formula provided above models generational growth, where there is one breeding time per year (or, at least a finite number); there is no explicit formula for this type of logistic growth.

Examples

A forest is currently home to a population of 200 rabbits. The forest is estimated to be able to sustain a population of 2000 rabbits. Absent any restrictions, the rabbits would grow by 50% per year. Predict the future population using the logistic growth model.

Answer

Modeling this with a logistic growth model, $r = 0.50$, $K = 2000$, and $P_0 = 200$. Calculating the next year:

$$P_1 = P_0 + 0.50 \left(1 - \frac{P_0}{2000}\right) P_0 = 200 + 0.50 \left(1 - \frac{200}{2000}\right) 200 = 290$$

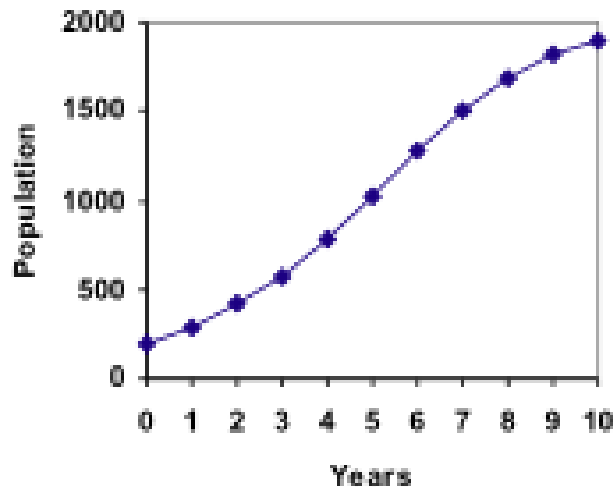
We can use this to calculate the following year:

$$P_2 = P_1 + 0.50 \left(1 - \frac{P_1}{2000}\right) P_1 = 290 + 0.50 \left(1 - \frac{290}{2000}\right) 290 \approx 414$$

A calculator was used to compute several more values:

n	0	1	2	3	4	5	6	7	8	9	10
P_n	200	290	414	578	784	1022	1272	1503	1690	1821	1902

Plotting these values, we can see that the population starts to increase faster and the graph curves upwards during the first few years, like exponential growth, but then the growth slows down as the population approaches the carrying capacity.



View more about this example below.

Watch this video online: <https://youtu.be/dPOIEgJ2QX0>

On an island that can support a population of 1000 lizards, there is currently a population of 600. These lizards have a lot of offspring and not a lot of natural predators, so have very high growth rate, around 150%. Calculating out the next couple generations:

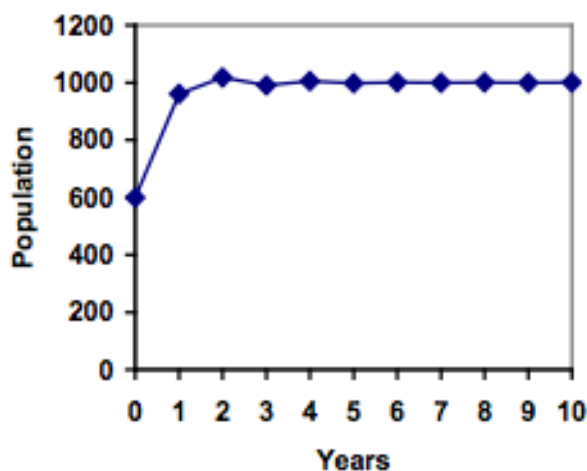
$$P_1 = P_0 + 1.50 \left(1 - \frac{P_0}{1000}\right) P_0 = 600 + 1.50 \left(1 - \frac{600}{1000}\right) 600 = 960$$

$$P_2 = P_1 + 1.50 \left(1 - \frac{P_1}{1000}\right) P_1 = 960 + 1.50 \left(1 - \frac{960}{1000}\right) 960 = 1018$$

Interestingly, even though the factor that limits the growth rate slowed the growth a lot, the population still overshoot the carrying capacity. We would expect the population to decline the next year.

$$P_3 = P_2 + 1.50 \left(1 - \frac{P_2}{1000}\right) P_2 = 1018 + 1.50 \left(1 - \frac{1018}{1000}\right) 1018 = 991$$

Calculating out a few more years and plotting the results, we see the population wavers above and below the carrying capacity, but eventually settles down, leaving a steady population near the carrying capacity.



Try It Now

A field currently contains 20 mint plants. Absent constraints, the number of plants would increase by 70% each year, but the field can only support a maximum population of 300 plants. Use the logistic model to predict the population in the next three years.

Answer

$$P_1 = P_0 + 0.70\left(1 - \frac{P_0}{300}\right)P_0 = 20 + 0.70\left(1 - \frac{20}{300}\right)20 = 33$$

$$P_2 = 54$$

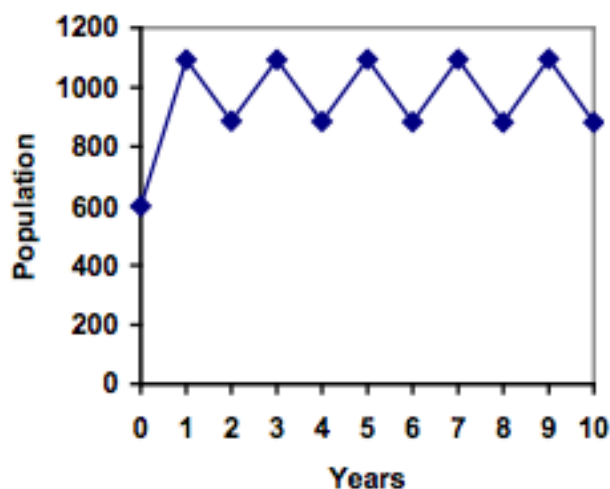
$$P_3 = 85$$

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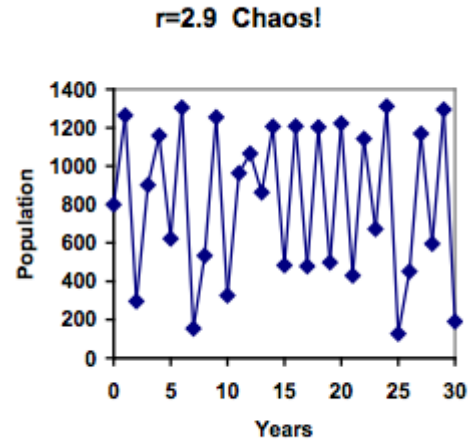
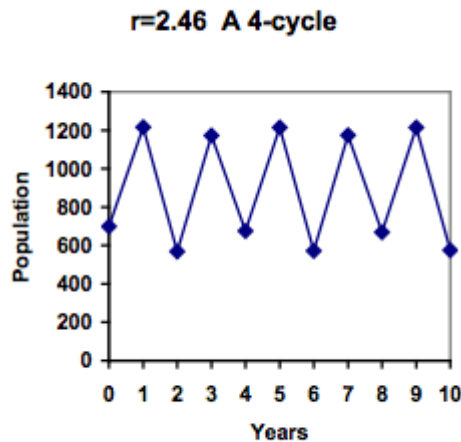
Example

On a neighboring island to the one from the previous example, there is another population of lizards, but the growth rate is even higher – about 205%.

Calculating out several generations and plotting the results, we get a surprise: the population seems to be oscillating between two values, a pattern called a 2-cycle.



While it would be tempting to treat this only as a strange side effect of mathematics, this has actually been observed in nature. Researchers from the University of California observed a stable 2-cycle in a lizard population in California. (Note: <http://users.rcn.com/jkimball.ma.ultranet/BiologyPages/P/Populations2.html>) Taking this even further, we get more and more extreme behaviors as the growth rate increases higher. It is possible to get stable 4-cycles, 8-cycles, and higher. Quickly, though, the behavior approaches chaos (remember the movie *Jurassic Park*?).



All of the lizard island examples are discussed in this video.
 Watch this video online: https://youtu.be/fuJF_uZGoFc

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PUTTING IT TOGETHER: GROWTH MODELS

At the beginning of the module, you were introduced to some social media statistics for several celebrities. Since then you learned that you could describe and compare growth by understanding a little bit about different types of growth, and using a growth model to make a prediction.

Now how about you be the one tracking the celebrities? Imagine that you started a celebrity blog a few years back. You write about hot news in Hollywood, and what your favorite celebs are up to. Between 2010 and 2012, your blog followers grew at a rate of 5% to 9,500 people. You hope to have 20,000 followers by 2020. If this growth rate continues, will you meet your goal?



Because you know that the number of followers is growing at 5%, you know you're dealing with exponential growth. In order to develop a model, you need an initial value. To determine the initial value, you can use the number of followers in 2012 ($P_0 = 9,500$), and the rate of growth, which is 5% ($r = 5\%$), and solve for P_0 .

$P_n = (1 + r)^n P_0$	Start with the explicit form of the exponential growth equation.
$P_n = (1 + 0.05)^n P_0$	Substitute the growth rate, 5%, written as a decimal for r .
$P_n = (1 + 0.05)^n 9,500$	Substitute the initial value, which is the number of followers in 2012, for P_0 .
$P_8 = (1 + 0.05)^8 9,500$	Substitute the number of years between 2012 and 2020 for n .
$P_8 = (1.05)^8 9,500 \approx 14,000$	Evaluate.

Unfortunately, you won't quite meet your goal, but you'll be close.

While online, you happen to notice that a different celebrity blogger is experiencing amazing success. She appeared on the scene only nine months ago, but in that short time it grew exponentially from an initial following of 5,000 people to 12,000. At what rate of growth is her following increasing?

$P_0 = 5,000$	Determine the initial value.
$P_9 = (1 + r)^9 P_0$	Again use the explicit form of the exponential growth equation, but this time $t = 9$ months.
$12,000 = (1 + r)^9 5,000$	Substitute the initial value, which is the number of followers when she started, for P_0 .
$\frac{12,000}{5,000} = (1 + r)^9$	Divide both sides by 5,000.

$\sqrt[9]{12,000/5,000} = (1 + r)^9$	Take the ninth root of both sides.
$r = \sqrt[9]{12,000/5,000} - 1 \approx 0.10$	Solve for r .

So her blog is growing at a rate of 10%. What's she got that you don't have? By comparing the rates of growth, you know it is definitely something. Now you just have to figure out what it is and jazz up your blog to get followers breaking down the internet to read what you have to say!

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FINANCE

WHY IT MATTERS: FINANCE

Why study interest formulas?

You're in the market for a new refrigerator, but don't have a lot of cash on hand to make the purchase. A flyer from an appliance rent-to-own store arrives in the mail one day, containing a very tempting offer: the refrigerator of your dreams for only \$17.99 per week!



The thought of paying \$17.99 a week seems reasonable given your current budget, but you hesitate when you read the fine print. The rent-to-own contract specifies that payments must be made for two full years. That's 104 weeks at \$17.99 per week!

At the local big box store, the same refrigerator is listed at only \$1299, including all taxes and fees. When you tell your brother about the two deals, he offers to help you buy the refrigerator from the big box store at the lower price of \$1299. However he will charge you 20% interest on the full price and wants you to pay off the balance within 12 months. You like the lower price, but 20% seems like a pretty high percentage to pay out to your brother.

Then you discover a third option. The big box store offers a store credit line at 15% APR. After reading the fine print, you learn that the credit line works just like a loan. The interest will be compounded each month, and there will be a fixed monthly payment for a total of 36 months. You wonder how much interest will accumulate on the \$1299 ticket price of the refrigerator.

Which offer is better? Renting-to-own for two years, buying it on a 20% loan from your brother, or using the store's line of credit at 15% compounding interest? Better think quickly: your ice cream is melting!

In order to make an informed decision, you will need to know the total cost for all three scenarios. The rent-to-own situation is the easiest to calculate because all of the fees and interest have been figured into the monthly payment already. Simply multiply the number of weeks in two full years by the weekly payment.

$$104 \times 17.99 = 1870.96$$

The other two scenarios involve interest formulas. We will revisit this scenario to see which offer is the best deal after taking a look at the other options.

In this module, you will learn two ways to calculate interest; simple and compound. Understanding interest rates will help you become a more informed consumer, potentially saving you a lot of money on big purchases such as appliances, cars and even your home.

Learning Objectives

Simple and Compound Interest

- Calculate future value and payments for savings annuities problems
- Calculate present value and payments for payout annuities problems
- Calculate present value and payments for loans problems

Annuities and Loans

- Determine the appropriate financial formula to use given a scenario by recognizing key words and examining frequency of deposits or withdrawals, and whether account is growing or decreasing in value
- Analyze a home mortgage refinance scenario, forming judgments by combining calculations and opinion
- Solve a financial application for time using logarithms

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INTRODUCTION: SIMPLE AND COMPOUND INTEREST

Learning Objectives

The learning objectives for this section include:

- Calculate one-time simple interest, and simple interest over time
- Determine APY given an interest scenario
- Calculate compound interest

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.



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SIMPLE INTEREST

Principal and Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the

principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = \$5$. The total amount you would repay would be \$105, the original principal plus the interest.



Simple One-time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0 (1 + r)$$

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: 5% = 0.05)

Examples

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Answer

the principal

$P_0 = \$300$	
$r = 0.03$	3% rate
$I = \$300(0.03) = \$9.$	You will earn \$9 interest.

The following video works through this example in detail.

Watch this video online: <https://youtu.be/TJYq7XGB8EY>

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly.

For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Exercises

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Answer

Each year, you would earn 5% interest: $\$1000(0.05) = \50 in interest. So over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1,000 you originally paid, leaving you with a total of \$1,250.

Further explanation about solving this example can be seen here.

Watch this video online: <https://youtu.be/rNOEYPCnGwg>

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = P_0 rt$$

$$A = P_0 + I = P_0 + P_0 rt = P_0 (1 + rt)$$

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up. For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
 $6 \div 12 = 0.5$
 A 4% annual rate paid quarterly would be divided into four 1% payments.
 $4 \div 4 = 1$

Example

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a \$1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Answer

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$	the principal
$r = 0.02$	2% rate per half-year
$t = 8$	4 years = 8 half-years
$I = \$1000(0.02)(8) = \$160.$	You will earn \$160 interest total over the four years.

This video explains the solution.

Watch this video online: <https://youtu.be/IfVn20go7-Y>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Try It Now

A loan company charges \$30 interest for a one month loan of \$500. Find the annual interest rate they are charging.

Answer

$I = \$30$ of interest

$P_0 = \$500$ principal

$r =$ unknown

$t = 1$ month

Using $I = P_0 r t$, we get $30 = 500 \cdot r \cdot 1$. Solving, we get $r = 0.06$, or 6%. Since the time was monthly, this is the monthly interest. The annual rate would be 12 times this: 72% interest.

Try It Now

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COMPOUND INTEREST

Compounding

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit \$1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3}{12} = 0.25\%$ per month.

In the first month,

- $P_0 = \$1000$
- $r = 0.0025$ (0.25%)
- $I = \$1000 (0.0025) = \2.50
- $A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn \$2.50 in interest, raising our account balance to \$1002.50.

In the second month,

- $P_0 = \$1002.50$
- $I = \$1002.50 (0.0025) = \2.51 (rounded)
- $A = \$1002.50 + \$2.51 = \$1005.01$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that **compounding** interest gives us.

Calculating out a few more months gives the following:



Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

We want to simplify the process for calculating compounding, because creating a table like the one above is time consuming. Luckily, math is good at giving you ways to take shortcuts. To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$$P_0 = \$1000$$

$$P_m = (1+0.0025)P_{m-1}$$

You probably recognize this as the recursive form of exponential growth. If not, we go through the steps to build an explicit equation for the growth in the next example.

Example

Build an explicit equation for the growth of \$1000 deposited in a bank account offering 3% interest, compounded monthly.

Answer

- $P_0 = \$1000$
- $P_1 = 1.0025P_0 = 1.0025(1000)$
- $P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$
- $P_3 = 1.0025P_2 = 1.0025(1.0025^2(1000)) = 1.0025^3(1000)$
- $P_4 = 1.0025P_3 = 1.0025(1.0025^3(1000)) = 1.0025^4(1000)$

Observing a pattern, we could conclude

- $P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0 , the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$P_m = P_0 \left(1 + \frac{r}{k}\right)^m$$

In this formula:

- m is the number of compounding periods (months in our example)
- r is the annual interest rate
- k is the number of compounds per year.

View this video for a walkthrough of the concept of compound interest.

Watch this video online: <https://youtu.be/xuQTFmP9nNg>

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0 \left(1 + \frac{r}{k}\right)^{Nk}$$

- P_N is the balance in the account after N years.
- P_0 is the starting balance of the account (also called initial deposit, or principal)
- r is the annual interest rate in decimal form
- k is the number of compounding periods in one year
 - If the compounding is done annually (once a year), $k = 1$.
 - If the compounding is done quarterly, $k = 4$.
 - If the compounding is done monthly, $k = 12$.
 - If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

In the next example, we show how to use the compound interest formula to find the balance on a certificate of deposit after 20 years.

Example

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Answer

In this example,

$P_0 = \$3000$	the initial deposit
----------------	---------------------

$r = 0.06$	6% annual rate
$k = 12$	12 months in 1 year
$N = 20$	since we're looking for how much we'll have after 20 years

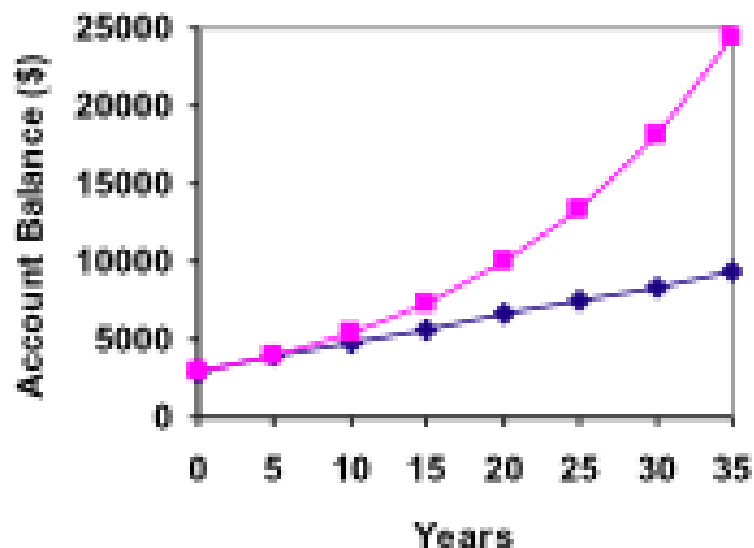
So $P_{20} = 3000 \left(1 + \frac{0.06}{12} \right)^{20 \times 12} = \9930.61 (round your answer to the nearest penny)

A video walkthrough of this example problem is available below.

Watch this video online: <https://youtu.be/8NazxAjhpJw>

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest (\$15 per month)	6% compounded monthly = 0.5% each month.
5	\$3900	\$4046.55
10	\$4800	\$5458.19
15	\$5700	\$7362.28
20	\$6600	\$9930.61
25	\$7500	\$13394.91
30	\$8400	\$18067.73
35	\$9300	\$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240} , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

\wedge , yx , or xy .

To evaluate 1.005^{240} we'd type $1.005 \wedge 240$, or $1.005 yx 240$. Try it out – you should get something around 3.3102044758.

Example

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Answer

In this example, we're looking for P_0 .

$r = 0.04$	4%
$k = 4$	4 quarters in 1 year
$N = 18$	Since we know the balance in 18 years
$P_{18} = \$40,000$	The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0 .

$$40000 = P_0 \left(1 + \frac{0.04}{4} \right)^{18 \times 4}$$

$$40000 = P_0 (2.0471)$$

$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a “close enough” answer, but keeping more digits is always better.

Example

To see why not over-rounding is so important, suppose you were investing \$1000 at 5% interest compounded monthly for 30 years.

$P_0 = \$1000$	the initial deposit
$r = 0.05$	5%
$k = 12$	12 months in 1 year
$N = 30$	since we're looking for the amount after 30 years

If we first compute r/k , we find $0.05/12 = 0.0041666666667$
Here is the effect of rounding this to different values:

r/k rounded to:	Gives P_{30} to be:	Error
0.004	\$4208.59	\$259.15
0.0042	\$4521.45	\$53.71
0.00417	\$4473.09	\$5.35
0.004167	\$4468.28	\$0.54
0.0041667	\$4467.80	\$0.06
no rounding	\$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough – \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

View the following for a demonstration of this example.

Watch this video online: <https://youtu.be/VhhYtaMN6mo>

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate $P_{30} = 1000 \left(1 + \frac{0.05}{12}\right)^{12 \times 30}$

We can quickly calculate $12 \times 30 = 360$, giving $P_{30} = 1000 \left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this	Calculator shows
$0.05 \div 12 = .$	0.00416666666667
$+ 1 = .$	1.00416666666667
$\text{yx } 360 = .$	4.46774431400613
$\times 1000 = .$	4467.74431400613

Using your calculator continued

The previous steps were assuming you have a “one operation at a time” calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

$$1000 \times (1 + 0.05 \div 12)^{\text{yx } 360} =$$

Solving For Time

Note: This section assumes you’ve covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Often we are interested in how long it will take to accumulate money or how long we’d need to extend a loan to bring payments down to a reasonable level.

Examples

If you invest \$2000 at 6% compounded monthly, how long will it take the account to double in value?

Answer

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$P_0 = \$2000$	the initial deposit
----------------	---------------------

$r = 0.06$	6% annual rate
$k = 12$	12 months in 1 year

So our general equation is $P_N = 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12}$. We also know that we want our ending amount to be double of \$2000, which is \$4000, so we're looking for N so that $P_N = 4000$. To solve this, we set our equation for P_N equal to 4000.

$$4000 = 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12}$$

Divide both sides by 2000

$$2 = (1.005)^{12N}$$

To solve for the exponent, take the log of both sides

$$\log(2) = \log\left((1.005)^{12N}\right)$$

Use the exponent property of logs on the right side

$$\log(2) = 12N \log(1.005)$$

Now we can divide both sides by $12 \log(1.005)$

$$\frac{\log(2)}{12 \log(1.005)} = N$$

Approximating this to a decimal

$$N = 11.581$$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded $\log(2)$ to 0.301 and $\log(1.005)$ to 0.00217, then your final answer would have been about 11.577 years.

Get additional guidance for this example in the following:

Watch this video online: <https://youtu.be/zHRTxtFiyxc>

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INTRODUCTION: ANNUITIES AND LOANS

Learning Objectives

The learning outcomes for this section include:

- Calculate the balance on an annuity after a specific amount of time
- Discern between compound interest, annuity, and payout annuity given a finance scenario
- Use the loan formula to calculate loan payments, loan balance, or interest accrued on a loan
- Determine which equation to use for a given scenario
- Solve a financial application for time

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. In this section, we will explore the math behind specific kinds of accounts that gain interest over time, like retirement accounts. We will also explore how mortgages and car loans, called installment loans, are calculated.



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SAVINGS ANNUITIES

Savings Annuity

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d , to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.



Example

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. Write an explicit

formula that represents this scenario.

Answer

In this example:

- $r = 0.06$ (6%)
- $k = 12$ (12 compounds/deposits per year)
- $d = \$100$ (our deposit per month)

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005) P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$

$$P_1 = (1.005)P_0 + 100 = 100$$

$$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$$

$$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after m deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100$$

In other words, after m months, the first deposit will have earned compound interest for $m-1$ months. The second deposit will have earned interest for $m-2$ months. The last month's deposit (L) would have earned only one month's worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005 \left(100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100 \right)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{array}{rcl} 1.005P_m & = & 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005) \\ P_m & = & 100(1.005)^{m-1} + \dots + 100(1.005) + 100 \end{array}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Factor P_m out of the terms on the left side.

$$P_m(1.005 - 1) = 100(1.005)^m - 100$$

$$(0.005)P_m = 100(1.005)^m - 100$$

Solve for P_m

$$0.005P_m = 100((1.005)^m - 1)$$

$$P_m = \frac{100((1.005)^m - 1)}{0.005}$$

Replacing m months with $12N$, where N is measured in years, gives

$$P_N = \frac{100((1.005)^{12N} - 1)}{0.005}$$

Recall 0.005 was r/k and 100 was the deposit d . 12 was k , the number of deposit each year.

Generalizing this result, we get the savings annuity formula.

Annuity Formula

$$P_N = \frac{d \left(\left(1 + \frac{r}{k} \right)^{Nk} - 1 \right)}{\left(\frac{r}{k} \right)}$$

- P_N is the balance in the account after N years.
- d is the regular deposit (the amount you deposit each year, each month, etc.)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

- If you make your deposits every month, use monthly compounding, $k = 12$.
- If you make your deposits every year, use yearly compounding, $k = 1$.
- If you make your deposits every quarter, use quarterly compounding, $k = 4$.
- Etc.

When do you use this?

Annuities assume that you put money in the account **on a regular schedule** (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account **once** and let it sit there earning interest.

- Compound interest: One deposit
- Annuity: Many deposits.

Examples

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Answer

In this example,

$d = \$100$	the monthly deposit
$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly deposits, we'll compound monthly
$N = 20$	we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100 \left(\left(1 + \frac{0.06}{12} \right)^{20(12)} - 1 \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_{20} = \frac{100 \left((1.005)^{240} - 1 \right)}{(0.005)}$$

$$P_{20} = \frac{100 (3.310 - 1)}{(0.005)}$$

$$P_{20} = \frac{100 (2.310)}{(0.005)} = \$46200$$

The account will grow to \$46,200 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is \$46,200 - \$24,000 = \$22,200.

This example is explained in detail here.

Watch this video online: <https://youtu.be/quLg4bRpxPA>

Try It Now

A conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer

$d = \$5$ the daily deposit

$r = 0.03$ 3% annual rate

$k = 365$ since we're doing daily deposits, we'll compound daily

$N = 10$ we want the amount after 10 years

$$P_{10} = \frac{5 \left(\left(1 + \frac{0.03}{365} \right)^{365 \cdot 10} - 1 \right)}{\frac{0.03}{365}} = 21,282.07$$

Try It Now

Visit this page in your course online to practice before taking the quiz.

Financial planners typically recommend that you have a certain amount of savings upon retirement. If you know the future value of the account, you can solve for the monthly contribution amount that will give you the desired result. In the next example, we will show you how this works.

Example

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Answer

In this example, we're looking for d .

$r = 0.08$	8% annual rate
$k = 12$	since we're depositing monthly
$N = 30$	30 years
$P_{30} = \$200,000$	The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d .

$$200,000 = \frac{d \left(\left(1 + \frac{0.08}{12} \right)^{30(12)} - 1 \right)}{\left(\frac{0.08}{12} \right)}$$

$$200,000 = \frac{d \left((1.00667)^{360} - 1 \right)}{(0.00667)}$$

$$200,000 = d(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit \$134.09 each month to have \$200,000 in 30 years if your account earns 8% interest.

View the solving of this problem in the following video.

Watch this video online: <https://youtu.be/LB6pl7o0REc>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Solving For Time

We can solve the annuities formula for time, like we did the compounding interest formula, by using logarithms. In the next example we will work through how this is done.

Example

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

Answer

This is a savings annuity problem since we are making regular deposits into the account.

$d = \$100$	the monthly deposit
$r = 0.03$	3% annual rate
$k = 12$	since we're doing monthly deposits, we'll compound monthly

We don't know N , but we want P_N to be \$10,000.

Putting this into the equation:

$$10,000 = \frac{100 \left(\left(1 + \frac{0.03}{12} \right)^{N(12)} - 1 \right)}{\left(\frac{0.03}{12} \right)}$$

Simplifying the fractions a bit

$$10,000 = \frac{100 \left((1.0025)^{12N} - 1 \right)}{0.0025}$$

We want to isolate the exponential term, 1.0025^{12N} , so multiply both sides by 0.0025

$$25 = 100 \left((1.0025)^{12N} - 1 \right)$$

Divide both sides by 100

$$0.25 = (1.0025)^{12N} - 1$$

Add 1 to both sides

$$1.25 = (1.0025)^{12N}$$

Now take the log of both sides

$$\log(1.25) = \log \left((1.0025)^{12N} \right)$$

Use the exponent property of logs

$$\log(1.25) = 12N \log(1.0025)$$

Divide by $12 \log(1.0025)$

$$\frac{\log(1.25)}{12 \log(1.0025)} = N$$

Approximating to a decimal

$$N = 7.447 \text{ years}$$

It will take about 7.447 years to grow the account to \$10,000.

This example is demonstrated here:

Watch this video online: <https://youtu.be/F3QVyswCzRo>

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PAYOUT ANNUITIES

Removing Money from Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the

account earns interest. After a fixed amount of time, the account will end up empty.



Payout annuities are typically used after retirement. Perhaps you have saved \$500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d \left(1 - \left(1 + \frac{r}{k} \right)^{-Nk} \right)}{\left(\frac{r}{k} \right)}$$

- P_0 is the balance in the account at the beginning (starting amount, or principal).
- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- r is the annual interest rate (in decimal form. Example: 5% = 0.05)
- k is the number of compounding periods in one year.
- N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this?

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

- Compound interest: One deposit
- Annuity: Many deposits.
- Payout Annuity: Many withdrawals

Example

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Answer

In this example,

$d = \$1000$	the monthly withdrawal
$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly withdrawals, we'll compound monthly
$N = 20$	since were taking withdrawals for 20 years

We're looking for P_0 : how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-20(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_0 = \frac{1000 \times \left(1 - (1.005)^{-240} \right)}{(0.005)}$$

$$P_0 = \frac{1000 \times (1 - 0.302)}{(0.005)} = \$139,600$$

You will need to have \$139,600 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is \$240,000 – \$139,600 = \$100,400 in interest.

View more about this problem in this video.

Watch this video online: <https://youtu.be/HK2eRFH6-0U>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this (-) , some with +/- . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: $1.005 \wedge (-) 240$

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: $1.005 \times 240 (-) =$

Give it a try – you should get $1.005^{-240} = 0.302096$

Example

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Answer

In this example, we're looking for d .

$r = 0.08$	8% annual rate
$k = 12$	since we're withdrawing monthly
$N = 30$	30 years
$P_0 = \$500,000$	we are beginning with \$500,000

In this case, we're going to have to set up the equation, and solve for d .

$$500,000 = \frac{d \left(1 - \left(1 + \frac{0.08}{12} \right)^{-30(12)} \right)}{\left(\frac{0.08}{12} \right)}$$

$$500,000 = \frac{d \left(1 - (1.00667)^{-360} \right)}{(0.00667)}$$

$$500,000 = d(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw \$3,670.21 each month for 30 years.

A detailed walkthrough of this example can be viewed here.

Watch this video online: <https://youtu.be/XK7rA6pD4cl>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Try It Now

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer

d = unknown
 $r = 0.04$ 4% annual rate
 $k = 1$ since we're doing annual scholarships
 $N = 20$ 20 years
 $P_0 = 100,000$ we're starting with \$100,000

$$100,000 = \frac{d \left(1 - \left(1 + \frac{0.04}{1} \right)^{-20 \cdot 1} \right)}{\frac{0.04}{1}}$$

Solving for d gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

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LOANS

Conventional Loans

In the last section, you learned about payout annuities. In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.



One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d \left(1 - \left(1 + \frac{r}{k} \right)^{-Nk} \right)}{\left(\frac{r}{k} \right)}$$

- P_0 is the balance in the account at the beginning (the principal, or amount of the loan).
- d is your loan payment (your monthly payment, annual payment, etc)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.
- N is the length of the loan, in years.

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this?

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

- Compound interest: One deposit
- Annuity: Many deposits
- Payout Annuity: Many withdrawals
- Loans: Many payments

Example

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

Answer

In this example,

$d = \$200$	the monthly loan payment
$r = 0.03$	3% annual rate
$k = 12$	since we're doing monthly payments, we'll compound monthly
$N = 5$	since we're making monthly payments for 5 years

We're looking for P_0 , the starting amount of the loan.

$$P_0 = \frac{200 \left(1 - \left(1 + \frac{0.03}{12} \right)^{-5(12)} \right)}{\left(\frac{0.03}{12} \right)}$$

$$P_0 = \frac{200 \left(1 - (1.0025)^{-60} \right)}{(0.0025)}$$

$$P_0 = \frac{200 (1 - 0.861)}{(0.0025)} = \$11,120$$

You can afford a \$11,120 loan.

You will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying \$12,000 - \$11,120 = \$880 interest total.

Details of this example are examined in this video.

Watch this video online: <https://youtu.be/5NiNcdYytvY>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Answer

In this example, we're looking for d .

$r = 0.06$	6% annual rate
$k = 12$	since we're paying monthly
$N = 30$	30 years

$$P_0 = \$140,000$$

the starting loan amount

In this case, we're going to have to set up the equation, and solve for d .

$$140,000 = \frac{d \left(1 - \left(1 + \frac{0.06}{12} \right)^{-30(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$140,000 = \frac{d \left(1 - (1.005)^{-360} \right)}{(0.005)}$$

$$140,000 = d(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of \$839.37 per month for 30 years.

You're paying a total of \$302,173.20 to the loan company: \$839.37 per month for 360 months. You are paying a total of \$302,173.20 – \$140,000 = \$162,173.20 in interest over the life of the loan.

View more about this example here.

Watch this video online: <https://youtu.be/BYCECTyUc68>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Try It Now

Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer

$$\begin{array}{ll}
 d = & \text{unknown} \\
 r = 0.16 & \text{16\% annual rate} \\
 k = 12 & \text{since we're making monthly payments} \\
 N = 2 & \text{2 years to repay} \\
 P_0 = 3,000 & \text{we're starting with a \$3,000 loan}
 \end{array}$$

$$3000 = \frac{d \left(1 - \left(1 + \frac{0.16}{12} \right)^{-2 \cdot 12} \right)}{\frac{0.16}{12}}$$

$$3000 = 20.42d$$

Solve for d to get monthly payments of \$146.89

Two years to repay means \$146.89(24) = \$3525.36 in total payments. This means Janine will pay \$3525.36 – \$3000 = \$525.36 in interest.

Calculating the Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.



To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think “how much loan will these loan payments be able to pay off in the remaining time on the loan?”

Example

If a mortgage at a 6% interest rate has payments of \$1,000 a month, how much will the loan balance be 10 years from the end the loan?

Answer

To determine this, we are looking for the amount of the loan that can be paid off by \$1,000 a month payments in 10 years. In other words, we're looking for P_0 when

$d = \$1,000$	the monthly loan payment
$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly payments, we'll compound monthly
$N = 10$	since we're making monthly payments for 10 more years

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-10(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_0 = \frac{1000 \left(1 - (1.005)^{-120} \right)}{(0.005)}$$

$$P_0 = \frac{1000 (1 - 0.5496)}{(0.005)} = \$90,073.45$$

The loan balance with 10 years remaining on the loan will be \$90,073.45.

This example is explained in the following video:

Watch this video online: <https://youtu.be/fXLzeyCfAwE>

Often times answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the *remaining time* on the loan

Example

A couple purchases a home with a \$180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Answer

First we will calculate their monthly payments.

We're looking for d .

$r = 0.04$	4% annual rate
$k = 12$	since they're paying monthly
$N = 30$	30 years
$P_0 = \$180,000$	the starting loan amount

We set up the equation and solve for d .

$$180,000 = \frac{d \left(1 - \left(1 + \frac{0.04}{12} \right)^{-30(12)} \right)}{\left(\frac{0.04}{12} \right)}$$

$$180,000 = \frac{d \left(1 - (1.00333)^{-360} \right)}{(0.00333)}$$

$$180,000 = d(209.562)$$

$$d = \frac{180,000}{209.562} = \$858.93$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan

balance that will be paid off with the monthly payments over those 25 years.

$d = \$858.93$	the monthly loan payment we calculated above
$r = 0.04$	4% annual rate
$k = 12$	since they're doing monthly payments
$N = 25$	since they'd be making monthly payments for 25 more years

$$P_0 = \frac{858.93 \left(1 - \left(1 + \frac{0.04}{12} \right)^{-25(12)} \right)}{\left(\frac{0.04}{12} \right)}$$

$$P_0 = \frac{858.93 \left(1 - (1.00333)^{-300} \right)}{(0.00333)}$$

$$P_0 = \frac{858.93 (1 - 0.369)}{(0.00333)} = \$155,793.91$$

The loan balance after 5 years, with 25 years remaining on the loan, will be \$155,793.91.

Over that 5 years, the couple has paid off \$180,000 – \$155,793.91 = \$24,206.09 of the loan balance. They have paid a total of \$858.93 a month for 5 years (60 months), for a total of \$51,535.80, so \$51,535.80 – \$24,206.09 = \$27,329.71 of what they have paid so far has been interest.

More explanation of this example is available here:

Watch this video online: <https://youtu.be/-J1Ak2LLyRo>

Solving for Time

Recall that we have used logarithms to solve for time, since it is an exponent in interest calculations. We can apply the same idea to finding how long it will take to pay off a loan.

Try It Now

Joel is considering putting a \$1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of \$30 a month?

Answer

$d = \$30$ The monthly payments
 $r = 0.12$ 12% annual rate
 $k = 12$ since we're making monthly payments
 $P_0 = 1,000$ we're starting with a \$1,000 loan

We are solving for N , the time to pay off the loan

$$1000 = \frac{30 \left(1 - \left(1 + \frac{0.12}{12} \right)^{-N \cdot 12} \right)}{\frac{0.12}{12}}$$

Solving for N gives 3.396. It will take about 3.4 years to pay off the purchase.

FYI

Home loans are typically paid off through an amortization process, **amortization** refers to paying off a debt (often from a loan or mortgage) over time through regular payments. An **amortization schedule** is a **table** detailing each periodic payment on an **amortizing** loan as generated by an **amortization calculator**.

If you want to know more, click on the link below to view the website “How is an Amortization Schedule Calculated?” by MyAmortizationChart.com. This website provides a brief overlook of Amortization Schedules.

- [How is an Amortization Schedule Calculated?](#)

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WHICH FORMULA TO USE?



Now that we have surveyed the basic kinds of finance calculations that are used, it may not always be obvious which one to use when you are given a problem to solve. Here are some hints on deciding which equation to use, based on the wording of the problem.

Loans

The easiest types of problems to identify are loans. Loan problems almost always include words like **loan**, **amortize** (the fancy word for loans), **finance** (i.e. a car), or **mortgage** (a home loan). Look for words like monthly or annual payment.

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Loans Formula

$$P_0 = \frac{d \left(1 - \left(1 + \frac{r}{k} \right)^{-Nk} \right)}{\left(\frac{r}{k} \right)}$$

- P_0 is the balance in the account at the beginning (the principal, or amount of the loan).
- d is your loan payment (your monthly payment, annual payment, etc)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.
- N is the length of the loan, in years.

Interest-Bearing Accounts

Accounts that gain interest fall into two main categories. The first is on where you put money in an account once and let it sit, the other is where you make regular payments or withdrawals from the account as in a retirement account.

Interest

- If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a **compound interest** problem. Look for words like compounded, or APY. Compound interest assumes that you put money in the account **once** and let it sit there earning interest.

COMPOUND INTEREST

$$P_N = P_0 \left(1 + \frac{r}{k} \right)^{Nk}$$

- P_N is the balance in the account after N years.
- P_0 is the starting balance of the account (also called initial deposit, or principal)
- r is the annual interest rate in decimal form
- k is the number of compounding periods in one year
 - If the compounding is done annually (once a year), $k = 1$.
 - If the compounding is done quarterly, $k = 4$.
 - If the compounding is done monthly, $k = 12$.
 - If the compounding is done daily, $k = 365$.

- The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at **simple interest**.

SIMPLE INTEREST OVER TIME

$$I = P_0 r t$$

$$A = P_0 + I = P_0 + P_0 r t = P_0 (1 + r t)$$

- I is the interest
- A is the end amount: principal plus interest

- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

Annuities

- If you're putting money **into** the account on a regular basis (monthly/annually/quarterly) then you're looking at a **basic annuity** problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future. Annuities assume that you put money in the account **on a regular schedule** (every month, year, quarter, etc.) and let it sit there earning interest.

ANNUITY FORMULA

$$P_N = \frac{d \left(\left(1 + \frac{r}{k} \right)^{Nk} - 1 \right)}{\left(\frac{r}{k} \right)}$$

- P_N is the balance in the account after N years.
- d is the regular deposit (the amount you deposit each year, each month, etc.)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

- If you're pulling money **out** of the account on a regular basis, then you're looking at a **payout annuity** problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future. Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

PAYOUT ANNUITY FORMULA

$$P_0 = \frac{d \left(1 - \left(1 + \frac{r}{k} \right)^{-Nk} \right)}{\left(\frac{r}{k} \right)}$$

- P_0 is the balance in the account at the beginning (starting amount, or principal).
- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- r is the annual interest rate (in decimal form. Example: 5% = 0.05)
- k is the number of compounding periods in one year.
- N is the number of years we plan to take withdrawals

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and then determine what approach will best allow you to solve the problem.

Try It Now

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

1. Marcy received an inheritance of \$20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?

Answer

This is a payout annuity problem. She can pull out \$1833.60 a quarter.

2. Paul wants to buy a new car. Rather than take out a loan, he decides to save \$200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?

Answer

This is a savings annuity problem. He will have saved up \$7,524.11

3. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have \$30,000 in the account in 6 years. How much should Keisha deposit into the account?

Answer

This is compound interest problem. She would need to deposit \$22,386.46.

4. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of \$100, how much new equipment can she buy?

Answer

This is a loans problem. She can buy \$4,609.33 of new equipment

5. How much would you need to save every month in an account earning 4% interest to have \$5,000 saved up in two years?

Answer

This is a savings annuity problem. You would need to save \$200.46 each month

In the following video, we present more examples of how to use the language in the question to determine which type of equation to use to solve a finance problem.

Watch this video online: <https://youtu.be/V5oG7ILTECs>

In the next video example, we show how to solve a finance problem that has two stages, the first stage is a savings problem, and the second stage is a withdrawal problem.

Watch this video online: <https://youtu.be/CNkvwMuLuis>

Try It Now

Visit this page in your course online to practice before taking the quiz.

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PUTTING IT TOGETHER: FINANCE

Financing a Refrigerator: Three Scenarios

In the beginning of this module, we presented three options for buying a new refrigerator. In one scenario, you could rent to own, with the following terms \$17.99 per week for 2 years, which is 104 weeks. The total cost is:

$$104 \times 17.99 = 1870.96$$

Scenario two involved a loan of \$1299 from your brother at 20% interest for one full year. To calculate the total amount, use the **simple interest formula**,

$$I = P_0 rt$$

In this situation, the principle amount is $P_0 = 1299$, rate is $r = 20\% = 0.20$, and the time is $t = 1$ year. Therefore, the interest due to your brother would be:

$$I = 1299 \times 0.20 \times 1 = 259.80$$

Adding the interest back to the principle, the total cost of the refrigerator would amount to \$1558.80. That's quite a bit less than the \$1870.96 that the rent-to-own store would ultimately have received from you. But your brother wants the money in one year, so let's figure out what the weekly payment would be. Simply divide the total by 52 weeks.

$$1558.80 \div 52 = 29.98$$

This is a higher weekly payment than the rent-to-own store is offering, but if you can afford it, then you'll save money in the long run.

Finally, let's explore the third option. This time we use the **loans formula**,

$$P_0 = \frac{d \left(1 - \left(1 + \frac{r}{k} \right)^{-Nk} \right)}{\left(\frac{r}{k} \right)}$$

The principle is the same, $P_0 = 1299$, but the rate is now $r = 15\% = 0.15$. Because the compounding is monthly, we have $k = 12$. Finally, $N = 3$ represents the total number of years for the loan. We must solve for d .

$$1299 = \frac{d \left(1 - \left(1 + \frac{0.15}{12} \right)^{-3(12)} \right)}{\left(\frac{0.15}{12} \right)}$$

$$1299 = \frac{d \left(1 - (1.0125)^{-36} \right)}{0.0125}$$

$$1299 = \frac{d(0.36059)}{0.0125}$$

$$d = 1299 \times 0.0125 \div 0.36059 = 45.03$$



This calculation gives the monthly payment (since the compounding is monthly), $d = 45.03$. If we want to see how this compares against our previous scenarios, we can find an equivalent weekly payment. The best way to do this is to multiply d by 12 and then divide by 52. This gives a weekly payment of about $45.03 \times 12 \div 52 = 10.39$, by far the lowest weekly payment, but what is the total cost?

Finally, to calculate the total cost, multiply the monthly payment by the number of months in 3 years, that is, 36 months.

$$45.03 \times 36 = 1621.09$$

Option three, the line of store credit. This option seemed pretty good at first. However, because of the long loan period and compounding interest, the total cost is actually more than the \$1558.80 from scenario two.

Let's compare the details of each scenario shown in the table below. Note, the **total interest** is found by subtracting the list price of the refrigerator (\$1299) from the **total paid** amount.

	Rent to Own	Brother's Offer	Store Loan
Payments	\$17.99 per week	\$29.98 per week	\$45.03 per month (\$10.39 per week)
Length of	2 years	1 year	3 years

Term			
Total Paid	\$1870.96	\$1558.80	\$1621.09
Total Interest	\$571.96	\$259.80	\$322.09

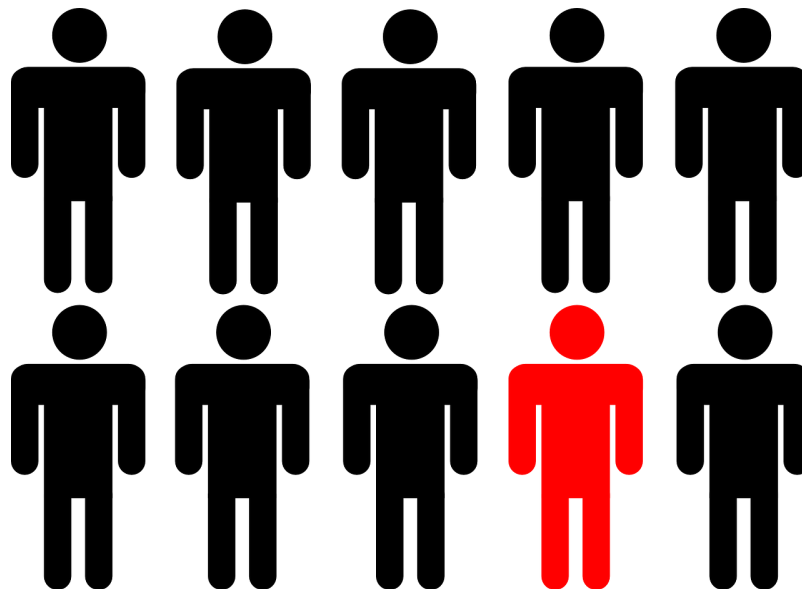
Which scenario would you choose?

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STATISTICS: COLLECTING DATA

WHY IT MATTERS: COLLECTING DATA

Like most people, you probably feel that it is important to “take control of your life.” But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, Statistics is one of the most important things that you can study.



To be more specific, here are some claims that we have heard on several occasions.

(We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.

- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.



Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, “There are three kinds of lies—lies, damned lies, and statistics.” This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950s and 60s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

Before we dive in, let's see a practical case of statistical analysis in action. You've likely seen several TED Talks videos at this point, either elsewhere in this course, in other courses you're taking, or out of personal interest. This amusing video analyzes the data that TED has gathered and produces some tips for how to put together the most (or least!) effective TED Talk possible.

Watch this video online: <https://youtu.be/1Totz8aa2Gg>

Learning Objectives

Data Collection

- Define the population and the parameters of a study
- Discern between a census and a population
- Define the sample and statistics of a study
- Classify data as categorical or quantitative
- Identify an appropriate sample for a study
- Identify possible sources of sampling bias
- Identify different techniques for sampling data

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INTRODUCTION: DATA COLLECTION BASICS

Learning Objectives

- Determine whether a value calculated from a group is a statistic or a parameter
- Identify the difference between a census and a sample
- Identify the population of a study
- Determine whether a measurement is categorical or qualitative

In this lesson we will introduce some important terminology related to collecting data. When you are finished you will be able to identify the difference between terms like census and sample. In the following lessons we will rely on your understanding of these terms, so study well!

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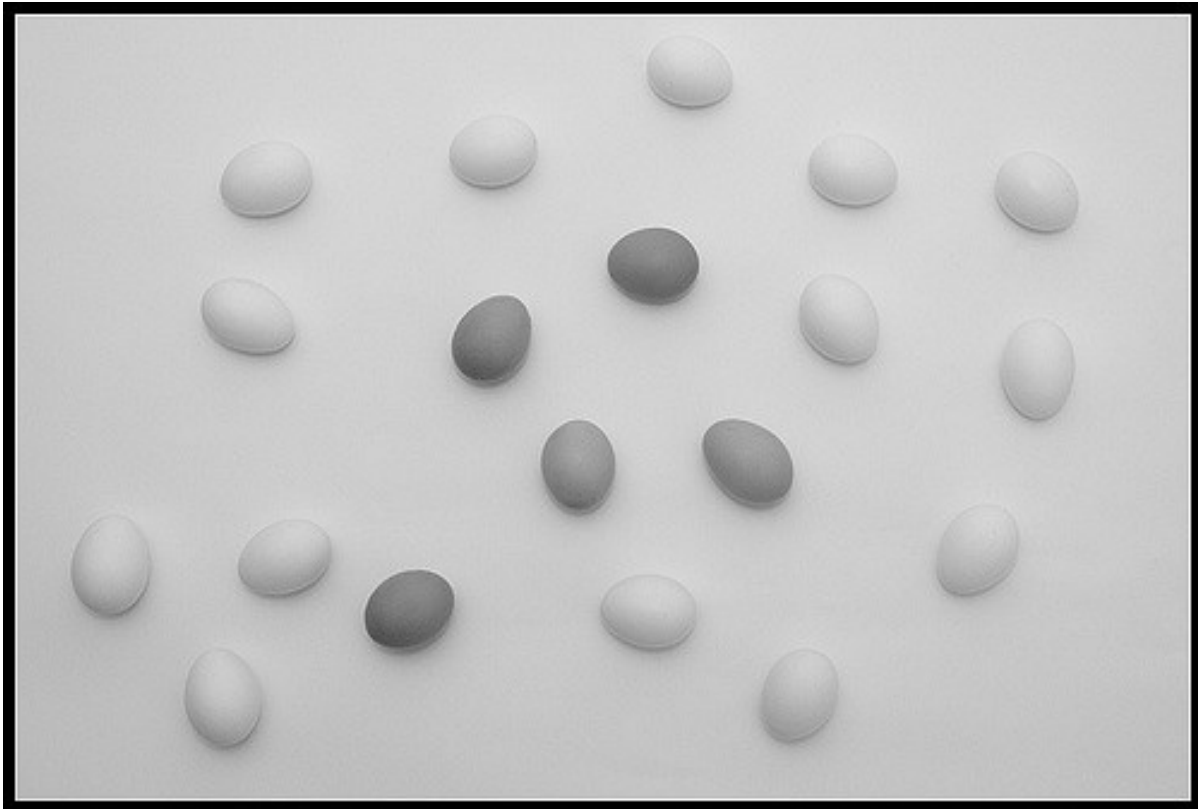
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POPULATIONS AND SAMPLES

Selecting A Focus

Before we begin gathering and analyzing data we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of Washington.
- All first-year students at public colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the entire United States.
- And so on.



Population

The **population** of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the **target population**, since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Washington might take slightly more diverse

courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from the University of Washington. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example

A newspaper website contains a poll asking people their opinion on a recent news article.
What is the population?

Answer

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define “average” more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Parameter

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

Census

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study.

Sample

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Statistic

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example

A researcher wanted to know how citizens of Tacoma felt about a voter initiative. To study this, she goes to the Tacoma Mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

Answer

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tacoma citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tacoma citizens.

The 60% value was based on the sample, so it is a statistic.

The examples on this page are detailed in the following video.

Watch this video online: <https://youtu.be/NlcDpqngBKY>

Try It Now

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Answer

The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.

Visit this page in your course online to practice before taking the quiz.

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

Answer

This is a parameter, since the college would have access to data on all students (the population)

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CATEGORIZING DATA

Quantitative or Categorical

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as categorical data or quantitative data.



Quantitative and categorical data

Categorical (qualitative) data are pieces of information that allow us to classify the objects under investigation into various categories.

Quantitative data are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like “*Finding Nemo*.” Is this categorical or quantitative data?

Answer

This would be categorical data.

Example

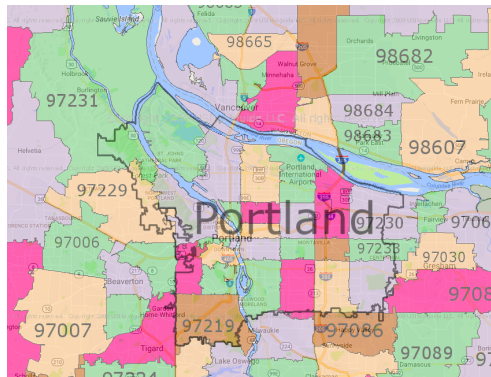
A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, . . .). Is this categorical or quantitative data?

Answer

This would be quantitative data. Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, . . .) or the amount of money you paid for a movie ticket the last time you went to a movie theater (\$5.50, \$7.75, \$9, . . .).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier. In the next example, the data collected is in numerical form, but it is not quantitative data. Read on to find out why.

Example



Zip Codes for Portland, OR

Suppose we gather respondents' ZIP codes in a survey to track their geographical location. Is this categorical or quantitative?

Answer

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

- 1 – it was awful
- 2 – it was just OK
- 3 – I liked it
- 4 – it was great
- 5 – best movie ever!

Is this categorical or quantitative?

Answer

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *Terminator 3*. And 33-37 year-olds might prefer . . . well, *Finding Nemo*.

The examples in this page are discussed further in the following video:

Watch this video online: <https://youtu.be/mxZqyB01qPY>

Try It Now

Classify each measurement as categorical or quantitative.

1. Eye color of a group of people
2. Daily high temperature of a city over several weeks
3. Annual income

Answer

1. Categorical. 2. Quantitative 3. Quantitative

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INTRODUCTION: SAMPLING AND EXPERIMENTATION

Learning Objectives

- Identify methods for obtaining a random sample of the intended population of a study
- Identify ineffective ways of obtaining a random sample from a population
- Identify types of sample bias
- Identify the differences between observational study and an experiment
- Identify the treatment in an experiment
- Determine whether an experiment may have been influenced by confounding

As we mentioned previously, the first thing we should do before conducting a survey is to identify the population that we want to study. In this lesson, we will show you examples of how to identify the population in a study, and determine whether or not the study actually represents the intended population. We will discuss different techniques for *random sampling* that are intended to ensure a population is well represented in a sample.

We will also identify the difference between an observational study and an experiment, and ways experiments can be conducted. By the end of this lesson, we hope that you will also be confident in identifying when an experiment may have been affected by confounding or the placebo effect, and the methods that are employed to avoid them.

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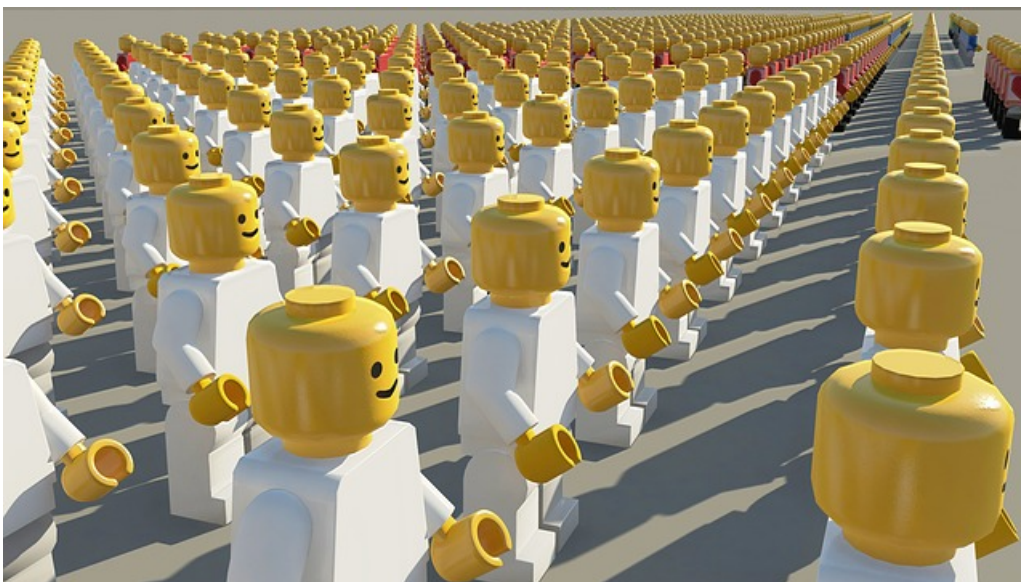
SAMPLING METHODS AND BIAS

Selecting a Population

Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote, and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?



In November 1998, former professional wrestler Jesse “The Body” Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

Sampling bias

A sampling method is biased if every member of the population doesn't have equal likelihood of being in the sample.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

Simple random sample

A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

Example

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample. In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the

state of Washington, some of these samples might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Sampling variability

The natural variation of samples is called **sampling variability**.
This is unavoidable and expected in random sampling, and in most cases is not an issue.

To help account for variability, pollsters might instead use a **stratified sample**.

Stratified sampling

In **stratified sampling**, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called **quota sampling**.

Quota sampling

Quota sampling is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat. You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any “real” questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling, but also does not ensure the same level of randomness.

Another sampling method is **cluster sampling**, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

Cluster sampling

In **cluster sampling**, the population is divided into subgroups (clusters), and a set of subgroups are selected to be in the sample.

Example

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

Systematic sampling

In **systematic sampling**, every n^{th} member of the population is selected to be in the sample.

Example

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book. Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample) but it can yield acceptable samples.

The Worst Way to Sample

Perhaps the worst types of sampling methods are **convenience samples** and **voluntary response samples**.

Convenience sampling and voluntary response sampling

Convenience sampling is the practice of samples chosen by selecting whoever is convenient.
Voluntary response sampling is allowing the sample to volunteer.

Example

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. Which sampling method is represented by this scenario?

Answer

This is a convenience sample.

A website has a survey asking readers to give their opinion on a tax proposal. Which sampling method is represented?

Answer

This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate. Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Watch the following video for an overview of all the sampling methods discussed so far.

Watch this video online: <https://youtu.be/sk4TDJU7QbY>

Try It Now

In each case, indicate what sampling method was used

- Every 4th person in the class was selected
- A sample was selected to contain 25 men and 35 women
- Viewers of a new show are asked to vote on the show's website
- A website randomly selects 50 of their customers to send a satisfaction survey to
- To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

Visit this page in your course online to practice before taking the quiz.

Problematic Sampling and Surveying

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – **sampling** or **selection bias**, which is when the sample is not representative of the population. One example of this is **voluntary response bias**, which is bias introduced by only collecting data from those who volunteer to participate. This is not the only potential source of bias.

Sources of bias

- **Sampling bias** – when the sample is not representative of the population
- **Voluntary response bias** – the sampling bias that often occurs when the sample is volunteers
- **Self-interest study** – bias that can occur when the researchers have an interest in the outcome
- **Response bias** – when the responder gives inaccurate responses for any reason
- **Perceived lack of anonymity** – when the responder fears giving an honest answer might negatively affect them
- **Loaded questions** – when the question wording influences the responses
- **Non-response bias** – when people refusing to participate in the study can influence the validity of the outcome

Example

Consider a recent study which found that chewing gum may raise math grades in teenagers (Note: Reuters. http://news.yahoo.com/s/nm/20090423/od_uk_nm/oukoe_uk_gum_learning. Retrieved 4/27/09). This study was conducted by the Wrigley Science Institute, a branch of the Wrigley chewing gum company. Identify the type of sampling bias found in this example.

Answer

This is an example of a **self-interest study**; one in which the researchers have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny.

A survey asks people “when was the last time you visited your doctor?” What type of sampling bias might this lead to?

Answer

This might suffer from **response bias**, since many people might not remember exactly when they last saw a doctor and give inaccurate responses. Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

A survey asks participants a question about their interactions with members of other races. Which sampling bias might occur for this survey strategy?

Answer

Here, a **perceived lack of anonymity** could influence the outcome. The respondent might not want to be perceived as racist even if they are, and give an untruthful answer.

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Which sampling bias may occur in this scenario?

Answer

Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer. This survey has the potential for **perceived lack of anonymity**.

A survey asks “do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?” Which sampling bias may result from this survey?

Answer

This is an example of a **loaded or leading question** – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda, or accidentally through poor question wording. Also a concern is **question order**, where the order of questions changes the results. A psychology researcher provides an example (Note: Swartz, Norbert. <http://www.umich.edu/~newsinfo/MT/01/Fal01/mt6f01.html>. Retrieved 3/31/2009):

“My favorite finding is this: we did a study where we asked students, ‘How satisfied are you with your life? How often do you have a date?’ The two answers were not statistically related – you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, ‘How often do you have a date? How satisfied are you with your life?’ the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student’s life as dating frequency.”

A telephone poll asks the question “Do you often have time to relax and read a book?”, and 50% of the people called refused to answer the survey. Which sampling bias is represented by this survey?

Answer

It is unlikely that the results will be representative of the entire population. This is an example of **non-response bias**, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

These problematic scenarios for statistics gathering are discussed further in the following video.

Watch this video online: <https://youtu.be/GXAi06PiVsk>

Try It Now

In each situation, identify a potential source of bias

- A survey asks how many sexual partners a person has had in the last year
- A radio station asks readers to phone in their choice in a daily poll.
- A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.
- High school students are asked if they have consumed alcohol in the last two weeks.
- The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
- A poll asks “Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?”

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EXPERIMENTS

Observing vs. Acting

So far, we have primarily discussed **observational studies** – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a “no turn on red” sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use **experiments** when exploring how subjects react to an outside influence. In an experiment, some kind of **treatment** is applied to the subjects and the results are measured and recorded.



Observational studies and experiments

- An **observational study** is a study based on observations or measurements
- An **experiment** is a study in which the effects of a **treatment** are measured

Examples

Here are some examples of experiments:

A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment here is the new drug.

A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment here is the new program.

You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.

A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

These examples are discussed further in the following video.

Watch this video online: <https://youtu.be/HSTTKzsdHEw>

Try It Now

Is each scenario describing an observational study or an experiment?

- a. The weights of 30 randomly selected people are measured
- b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured
- c. Twenty coffee drinkers and twenty tea drinkers are given a concentration test

Answer

- a. Observational study
- b. Experiment; the treatment is the jumping jacks
- c. Experiment; the treatments are coffee and tea

Visit this page in your course online to practice before taking the quiz.

When conducting experiments, it is essential to isolate the treatment being tested.

Example

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. They decide to run an experiment to see if an alternate curriculum would improve scores. To run the test, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement, or whether the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments, though sometimes it is hidden.

Confounding

Confounding occurs when there are two potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

Example

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. However, in the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both. In this case confounding has occurred.

Example

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music, then give a similar test while the student listens to music. In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

View the following for additional discussion of these examples.

Watch this video online: <https://youtu.be/SrCm12HZay0>

Try It Now

Visit this page in your course online to practice before taking the quiz.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a **control group**.

Control group

When using a control group, the participants are divided into two or more groups, typically a **control group** and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

Example

To determine if a two day prep course would help high school students improve their scores on the SAT test, a group of students was randomly divided into two subgroups. The first group, the treatment group, was given a two day prep course. The second group, the control group, was not given the prep course. Afterwards, both groups were given the SAT.

A company testing a new plant food grows two crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill, or a response to have taken any pill. This is called a **placebo effect**.

Placebo effect

The **placebo effect** is when the effectiveness of a treatment is influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

Example

A study found that when doing painful dental tooth extractions, patients told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine. (Note: Levine JD, Gordon NC, Smith R, Fields HL. (1981) Analgesic responses to morphine and placebo in individuals with postoperative pain. Pain. 10:379-89.)

To control for the placebo effect, a **placebo**, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

Placebo and Placebo controlled experiments

- A **placebo** is a dummy treatment given to control for the placebo effect.
- An experiment that gives the control group a placebo is called a **placebo controlled experiment**.

Example

In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.

In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.

In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

The following video walks through the controlled experiment scenarios, including the ones using placebos. Watch this video online: <https://youtu.be/UkCHUeqMb5Y>

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the second group, called a **comparison group** in this case. In our SAT test example, the non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

Blind studies

- A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo.

- A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the control group.

Example

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

These examples are demonstrated in the following video.

Watch this video online: <https://youtu.be/7BFZVGCxeYc>

Try It Now

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

Answer

The truth-telling group could be considered the control group, but really both groups are treatment groups here, since it is important for the lie detector to be able to correctly identify lies, and also not identify truth telling as lying. This study is blind, since the person running the test does not know what group each subject is in.

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PUTTING IT TOGETHER: COLLECTING DATA

You've been learning about ways to make sense of the overload of data that constantly surrounds you. Now that you've completed the module, you can better determine how to collect data and determine what the data truly represent.

At some point, you have probably been involved in some type of attempt to collect data. Perhaps you were asked to take a "quick" survey about your attitude toward some type of political or consumer issue, and you agreed.

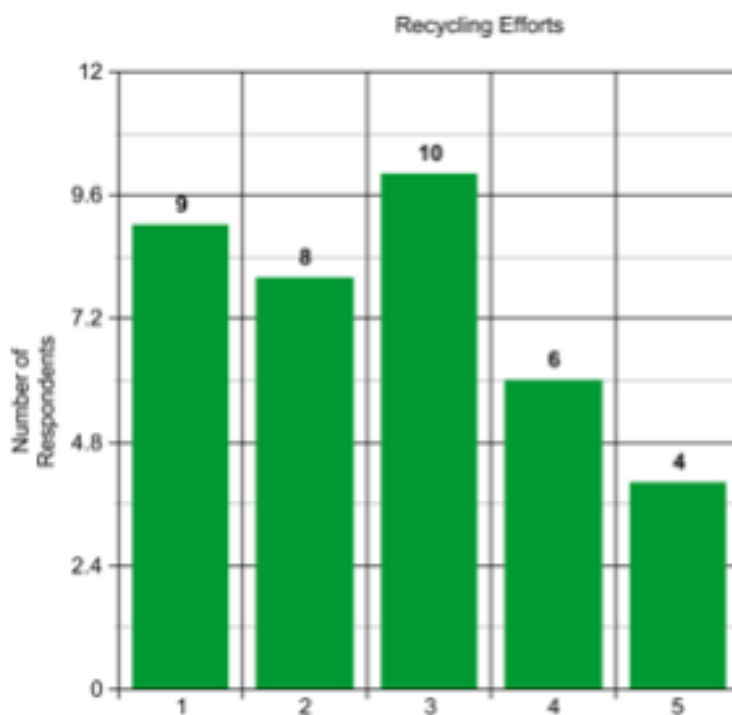


The survey took longer than promised, but at least you helped the person collect random and unbiased data. Or did you? In order to appreciate the data collected, you need to ask yourself a few questions:

- Is it possible to get a truly random sample from a phone survey?
- What is the population of the sample?
- What are possible sources of bias?
- Is the data being collected from you a statistic or a parameter? Is it categorical or qualitative?
- Is this an experiment or observational study?

Calling a list of phone numbers is partly random, but there is potential to leave out part of the population. For example, a phone survey misses people who don't have a landline and use only a cell phone. It also misses people who monitor calls and let the answering machine pick up. So the sample you were part of wasn't truly random.

Let's consider some of the results, which are mailed to you at a later date. The first thing you see is this bar graph. What does it suggest?



1= never, 5= as much as possible

When you look closely, you notice that the data does not indicate a specific quantitative amount, like how many times a week the respondent recycles. The survey instead takes qualitative data – self-reflection about likelihood to recycle – and quantifies that information on a scale of 1 to 5.

You can also determine that the majority of respondents think they recycle less than, or equal to average. It also shows that the category selected by the most survey-takers is 3, halfway between recycling as much as possible and never recycling. But is this what the general public truly thinks about recycling?

To find out, you need to take this research into your hands by considering additional methods of data collection. If you are interested in qualitative data, you must be prepared for a little more time-consuming research. The main methods of conducting this research involve individual interviews, focus groups, and direct observation. Maybe you might ask people at random around town. Or you might prefer quantitative data, and count the number of recycle bins placed outside homes on the proper day. But be careful not to bias your results. If you include data collected in a town that does not have recycling pickup, you will most likely obtain different results than in an area where recycling pickup is easy and free.

Collecting unbiased, useful data is a challenging task. You must always be sure to take possible errors into account and design your data collection method to minimize them. How would you design a method to collect data about society's dedication to recycling?

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STATISTICS: DESCRIBING DATA

WHY IT MATTERS: DESCRIBING DATA

The following video provides many examples of how contextualized, visually beautiful numbers add meaning and relevance to our life.

Watch this video online: <https://youtu.be/5Zg-C8AAIGg>

In the following pages, we'll explore how to make information meaningful in a visual format. This will help us solve some problems more quickly, and allow us better understanding for how to compare different types of information to make better decisions as a result.

Learning Objectives

Statistical Analysis

- Present categorical data graphically using a frequency table, bar graph, Pareto chart, pie charts, pictograms
- Present quantitative data graphically using histograms, frequency tables, pie charts, or frequency polygons
- Define the measures of central tendency for a sample of data including mean, median, mode
- Define measures of variation of a sample of data including range, standard deviation, quartiles, box plots

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- The beauty of data visualization - David McCandless. **Authored by:** TED-Ed. **Located at:** <https://youtu.be/5Zg-C8AAIGg>. **License:** All Rights Reserved. **License Terms:** Standard YouTube License

INTRODUCTION: REPRESENTING DATA GRAPHICALLY

Learning Objectives

- Create a frequency table, bar graph, pareto chart, pictogram, or a pie chart to represent a data set
- Identify features of ineffective representations of data
- Create a histogram, pie chart, or frequency polygon that represents numerical data
- Create a graph that compares two quantities

In this lesson we will present some of the most common ways data is represented graphically. We will also discuss some of the ways you can increase the accuracy and effectiveness of graphs of data that you create.

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PRESENTING CATEGORICAL DATA GRAPHICALLY

Visualizing Data

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table**.

Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To

research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Try It Now

Visit this page in your course online to practice before taking the quiz.

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

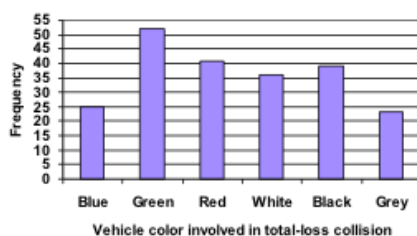
Bar graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

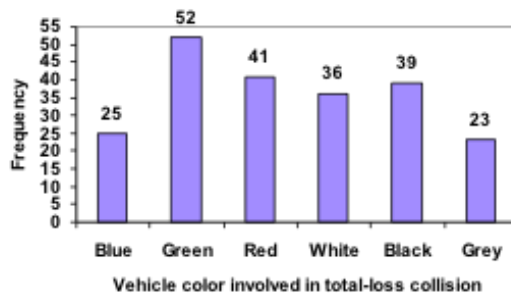
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by use of an example.

Example

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



The following video explains the process and value of moving data from a table to a bar graph. Watch this video online: https://youtu.be/vwxKf_O3ui0

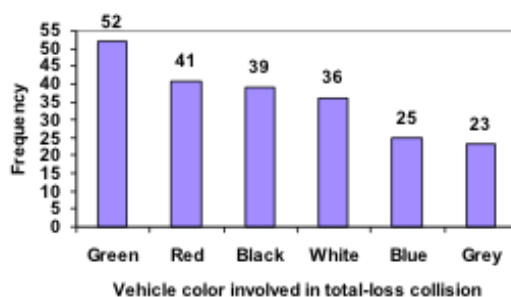
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Pareto chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Examples

Transforming our bar graph from earlier into a Pareto chart, we get:



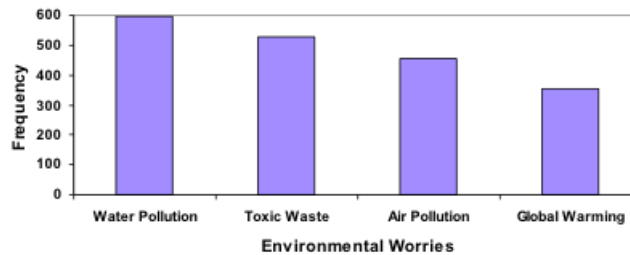
The following video addressed Pareto charts. Watch this video online: <https://youtu.be/Tsvru8DPxBE>

In a survey (Note: Gallup Poll. March 5-8, 2009. <http://www.pollingreport.com/enviro.htm>), adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried “a great deal” about some selected concerns are summarized below.

Environmental Issue	Frequency

Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:



To show relative sizes, it is common to use a pie chart.

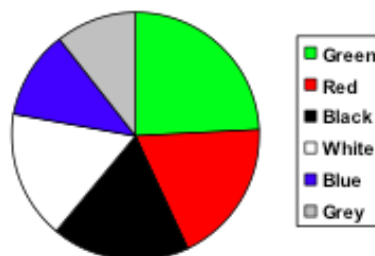
Pie Chart

A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example

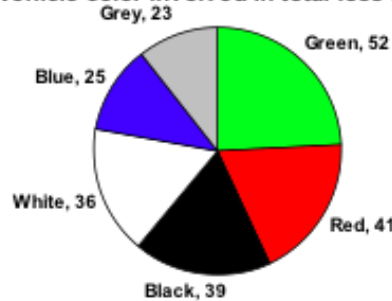
For our vehicle color data, a pie chart might look like this:

Vehicle color involved in total-loss collisions



Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.

Vehicle color involved in total-loss collisions



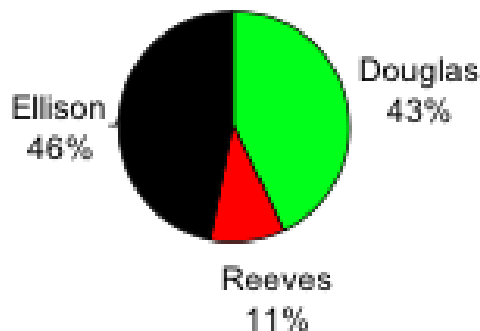
This video demonstrates how to **create** pie charts like the ones above.

Watch this video online: https://youtu.be/___1f8dKh6yo

The pie chart below shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

Voter preferences



The following video addresses how to **read** a pie chart like the one above.

Watch this video online: <https://youtu.be/mwa8vQnGr3I>

Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Drive are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs. (Note: For example: <http://nces.ed.gov/nceskids/createAgraph/> or <http://docs.google.com>)

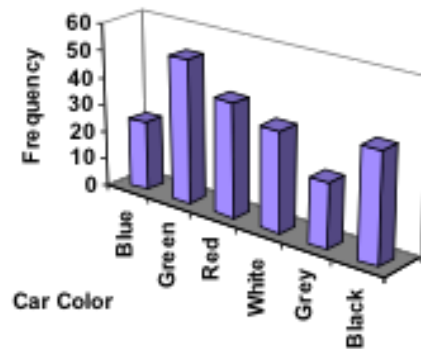
Try It Now

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Visit this page in your course online to practice before taking the quiz.

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.



Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.

Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example

A labor union might produce the graph to the right to show the difference between the average manager salary and the average worker salary.

Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

This video reviews the two examples of ineffective data representation in more detail.

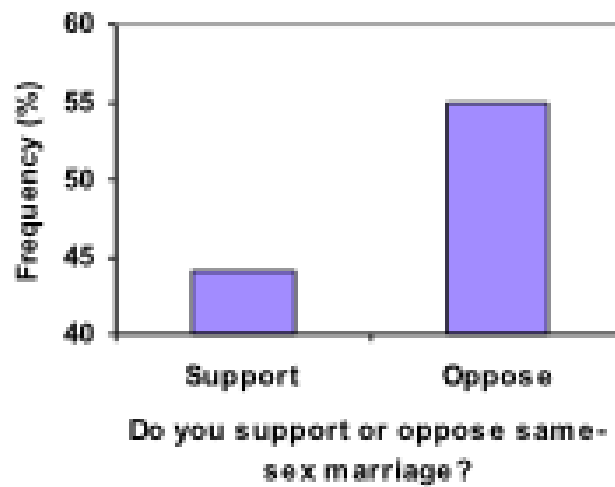
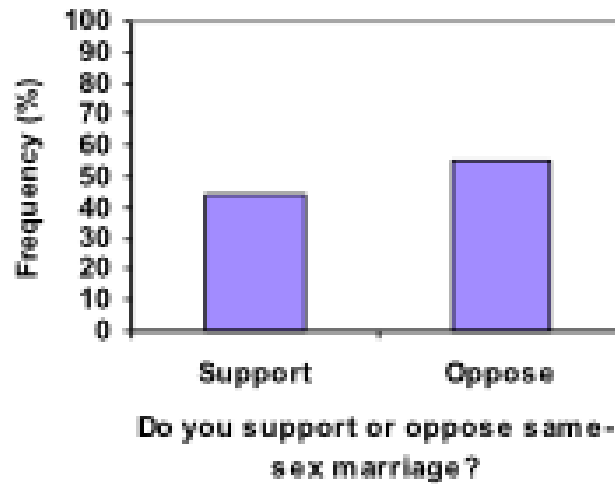
Watch this video online: <https://youtu.be/bFwTZNGNLKs>



Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

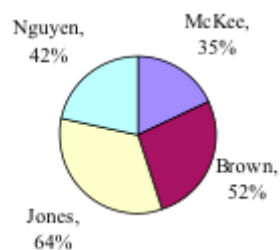
Example

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008 (Note: CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from <http://www.pollingreport.com/civil.htm>). The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.



Try It Now

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



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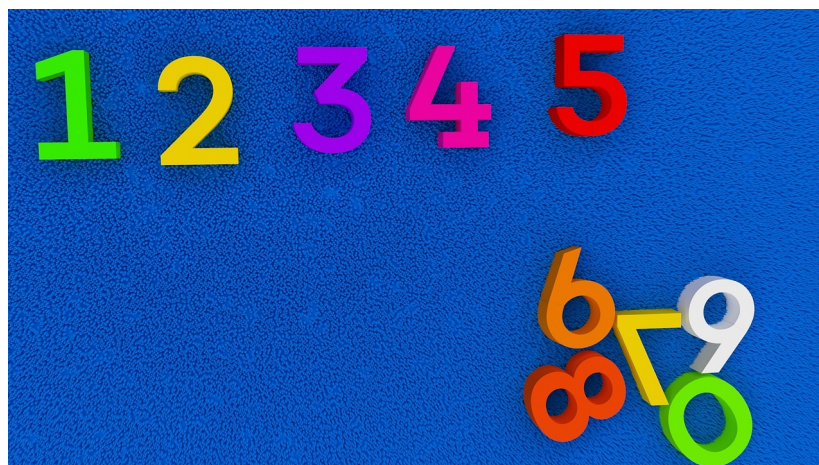
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PRESENTING QUANTITATIVE DATA GRAPHICALLY

Visualizing Numbers



Quantitative, or numerical, data can also be summarized into frequency tables.

Example

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

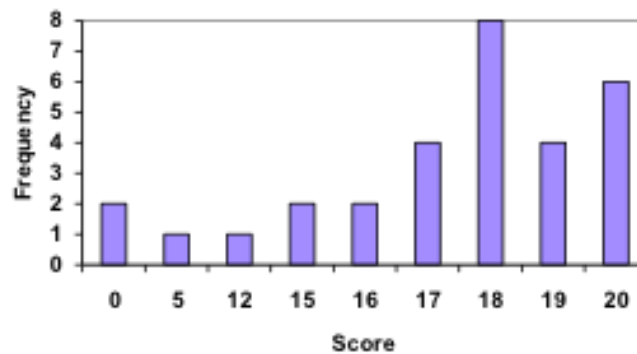
19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2

17	4
18	8
19	4
20	6

Using the table from the first example, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



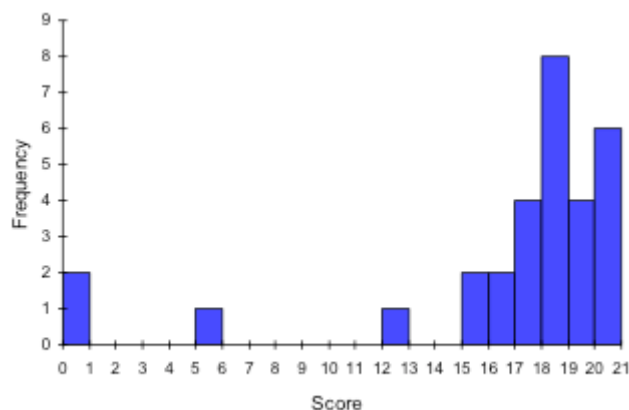
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram

A histogram is like a bar graph, but where the horizontal axis is a number line.

Example

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at $\frac{1}{2}$ values to avoid this ambiguity.

This video demonstrates the creation of the histogram from this data.

Watch this video online: https://youtu.be/180FgZ_cTrE

Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that

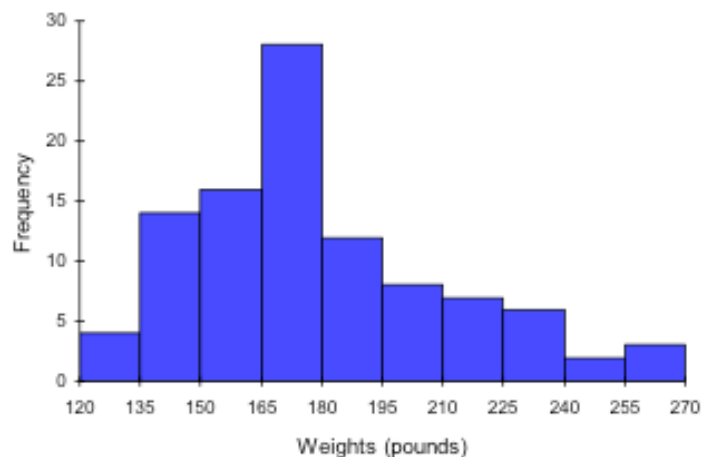
- each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- we have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example

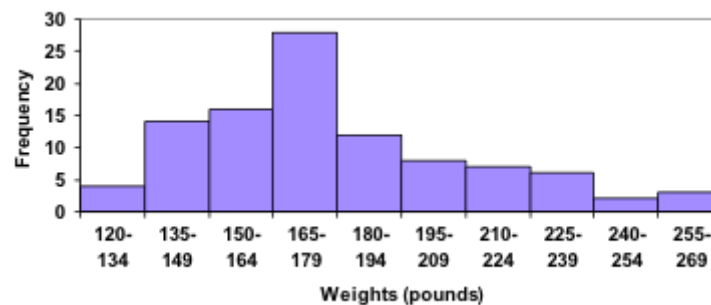
Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of $263 - 121 = 142$. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Interval	Frequency
120 – 134	4
135 – 149	14
150 – 164	16
165 – 179	28
180 – 194	12
195 – 209	8
210 – 224	7
225 – 239	6
240 – 254	2
255 – 269	3

A histogram of this data would look like:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.

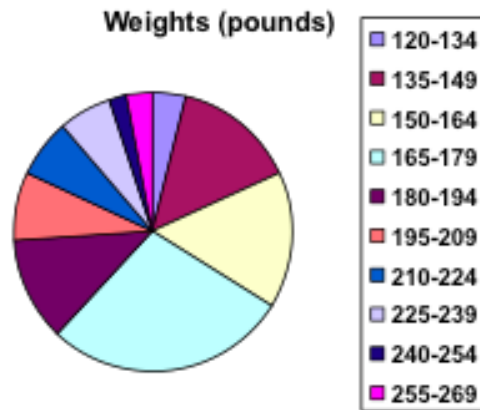


The following video walks through this example in more detail.
Watch this video online: <https://youtu.be/JhshiTtdP0>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



To see more about why a pie chart isn't useful in this case, watch the following.

Watch this video online: <https://youtu.be/FQ8zmZ56-XA>

Try It Now

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

\$140 \$160 \$160 \$165 \$180 \$220 \$235 \$240 \$250 \$260 \$280 \$285
 \$285 \$285 \$290 \$300 \$300 \$305 \$310 \$310 \$315 \$315 \$320 \$320
 \$330 \$340 \$345 \$350 \$355 \$360 \$360 \$380 \$395 \$420 \$460 \$460

When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

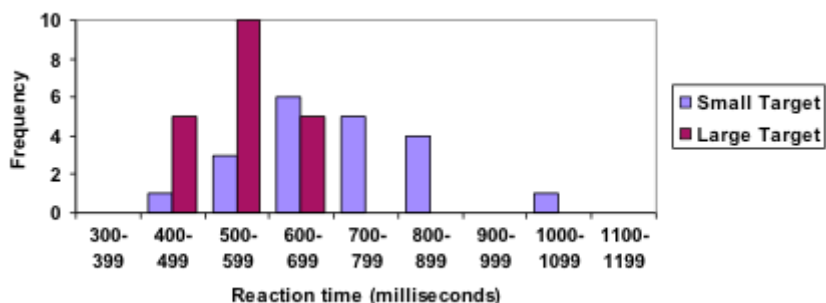
Interval (milliseconds)	Frequency small target	Frequency large target
300-399	0	0
400-499	1	5
500-599	3	10
600-699	6	5
700-799	5	0
800-899	4	0
900-999	0	0
1000-1099	1	0

1100-1199

0

0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

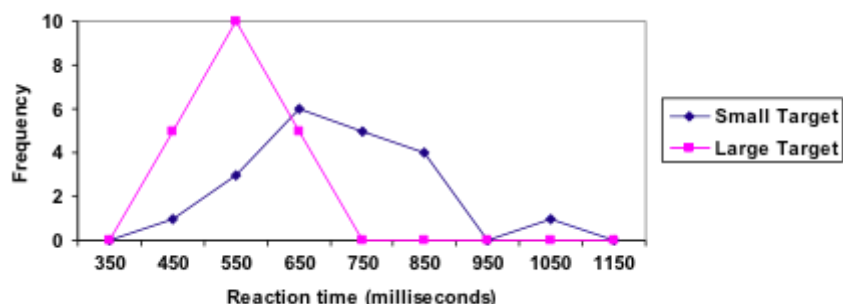


Frequency polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



The following video explains frequency polygon creation for this example.
Watch this video online: <https://youtu.be/rxByzA9MFFY>

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INTRODUCTION: NUMERICAL SUMMARIES OF DATA

Learning Objectives

- Calculate the mean, median, and mode of a set of data
- Calculate the range of a data set, and recognize its limitations in fully describing the behavior of a data set
- Calculate the standard deviation for a data set, and determine its units
- Identify the difference between population variance and sample variance
- Identify the quartiles for a data set, and the calculations used to define them
- Identify the parts of a five number summary for a set of data, and create a box plot using it

It is often desirable to use a few numbers to summarize a data set. One important aspect of a set of data is where its center is located. In this lesson, measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second section of this lesson describes measures of variability.

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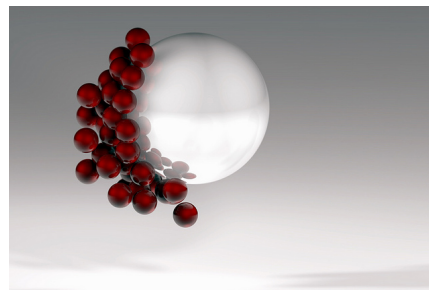
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MEASURES OF CENTRAL TENDENCY

Mean, Median, and Mode

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average."



Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example

Marci's exam scores for her last math class were 79, 86, 82, and 94. What would the mean of these values would be?

Answer

$\frac{79+86+82+94}{4} = 85.25$. Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

What is the mean number of TD passes?

Answer

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get $634/31 = 20.4516$. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Both examples are described further in the following video.

Watch this video online: <https://youtu.be/3if9Le2sO0c>

Try It Now

The price of a jar of peanut butter at 5 stores was \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the mean price.

Example

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest \$5 thousand dollars. The results are summarized in a frequency table below.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19

40	20
45	12
50	7

What is the mean average income in this neighborhood?

Answer

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\underbrace{15+\dots+15}_{6\text{terms}} + \underbrace{20+\dots+20}_{8\text{terms}} + \underbrace{25+\dots+25}_{11\text{terms}} + \dots$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $= 90$. Using this simplification, we get

$$\frac{15 \cdot 6 + 20 \cdot 8 + 25 \cdot 11 + 30 \cdot 17 + 35 \cdot 19 + 40 \cdot 20 + 45 \cdot 12 + 50 \cdot 7}{100} = \frac{3390}{100} = 33.9$$

The mean household income of our sample is 33.9 thousand dollars (\$33,900).

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of \$5 million (\$5000 thousand).

What is the new mean of this neighborhood's income?

Answer

Adding this to our sample, our mean is now:

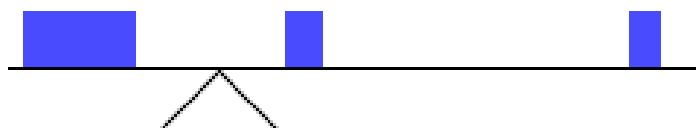
$$\frac{15 \cdot 6 + 20 \cdot 8 + 25 \cdot 11 + 30 \cdot 17 + 35 \cdot 19 + 40 \cdot 20 + 45 \cdot 12 + 50 \cdot 7 + 5000 \cdot 1}{101} = \frac{8390}{101} = 83.069$$

Both situations are explained further in this video.

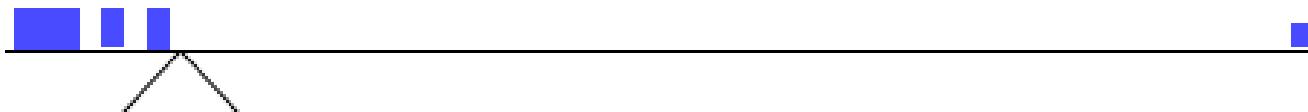
Watch this video online: https://youtu.be/1_4Hxcq8DpQ

While 83.1 thousand dollars (\$83,069) is the correct mean household income, it no longer represents a "typical" value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the \$5 million data value is so far out to the right that the mean has to adjust up to keep things in balance.



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order.

- To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.
- If the number of data values, N , is odd, then the median is the middle data value. This value can be found by rounding $N/2$ up to the next whole number.
- If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values $N/2$ and $N/2 + 1$)

Example

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 12 12 9 6

What is the median TD value?

Answer

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value ($31/2 = 15.5$, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

Answer

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get $(6+7)/2 = 6.5$.

The median quiz score was 6.5.

Learn more about these median examples in this video.

Watch this video online: https://youtu.be/WEdr_rSRObk

Try It Now

The price of a jar of peanut butter at 5 stores was \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the median price.

Example

Let us return now to our original household income data

Income (thousands of dollars)	Frequency
15	6

20	8
25	11
30	17
35	19
40	20
45	12
50	7

What is the mean of this neighborhood's household income?

Answer

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values – the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of \$15, so	Values 1 to 6 are \$15 thousand
The next 8 data values are \$20, so	Values 7 to $(6+8)=14$ are \$20 thousand
The next 11 data values are \$25, so	Values 15 to $(14+11)=25$ are \$25 thousand
The next 17 data values are \$30, so	Values 26 to $(25+17)=42$ are \$30 thousand
The next 19 data values are \$35, so	Values 43 to $(42+19)=61$ are \$35 thousand

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand.

If we add in the new neighbor with a \$5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

View more about the median of this neighborhood's household incomes here.

Watch this video online: <https://youtu.be/kqEu9EDkmfU>

In addition to the mean and the median, there is one other common measurement of the “typical” value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example

In our vehicle color survey earlier in this section, we collected the data

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

Which color is the mode?

Answer

For this data, Green is the mode, since it is the data value that occurred the most frequently.

Mode in this example is explained by the video here.

Watch this video online: <https://youtu.be/pFpkWrib3Jk>

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try It Now

Reviewers were asked to rate a product on a scale of 1 to 5. Find

1. The mean rating
2. The median rating
3. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

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MEASURES OF VARIATION

Range and Standard Deviation

Consider these three sets of quiz scores:

Section A: 5 5 5 5 5 5 5 5 5

Section B: 0 0 0 0 0 10 10 10 10 10

Section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the “typical” or “middle” value, we also need a measure of how “spread out” or varied each data set is.



There are several ways to measure this “spread” of the data. The first is the simplest and is called the **range**.

Range

The range is the difference between the maximum value and the minimum value of the data set.

Example

Using the quiz scores from above,

For section A, the range is 0 since both maximum and minimum are 5 and $5 - 5 = 0$

For section B, the range is 10 since $10 - 0 = 10$

For section C, the range is 2 since $6 - 4 = 2$

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

The range of this example is explained in the following video.

Watch this video online: <https://youtu.be/b3ofWalrHgQ>

Standard deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value – mean
0	$0 - 5 = -5$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$

5	5-5 = 0
10	10-5 = 5

We would like to get an idea of the “average” deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

data value	deviation: data value – mean	deviation squared
0	0-5 = -5	$(-5)^2 = 25$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
10	10-5 = 5	$(5)^2 = 25$

We then add the squared deviations up to get $25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50$. Ordinarily we would then divide by the number of scores, n , (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n - 1$ (in this case, $10 - 1 = 9$). (Note: The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.)

So in our example, we would have $50/10 = 5$ if section D represents a population and $50/9 =$ about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{population standard deviation} = \sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$$

or

$$\text{sample standard deviation} = \sqrt{\frac{50}{9}} \approx 2.4$$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by n , the number of data values, if the data represents a whole population; divide by $n - 1$ if the data is from a sample.
5. Compute the square root of the result.

Example

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value – mean	deviation squared
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8

Section D: 0 5 5 5 5 5 5 10

Standard deviation: 2.2

See the following video for more about calculating the standard deviation in this example.
Watch this video online: <https://youtu.be/wS8z90f04OU>

Try It Now

The price of a jar of peanut butter at 5 stores was \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q1) is the value so that 25% of the data values are below it; the third quartile (Q3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q1, 25% is between Q1 and the median, 25% is between the median and Q3, and 25% is between Q3 and the maximum value.

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:

Minimum, Q1, Median, Q3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n . If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q1

1. Begin by ordering the data from smallest to largest
2. Compute the locator: $L = 0.25n$
3. If L is a decimal value:
 - Round up to $L+$
 - Use the data value in the $L+$ th position
4. If L is a whole number:
 - Find the mean of the data values in the L th and $L+1$ th positions.

To find the third quartile, Q3

Use the same procedure as for Q1, but with locator: $L = 0.75n$

Examples should help make this clearer.

Example

Suppose we have measured 9 females, and their heights (in inches) sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

What are the first and third quartiles?

Answer

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is $0.75(9) = 6.75$. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Suppose we had measured 8 females, and their heights (in inches) sorted from smallest to largest are:

59 60 62 64 66 67 69 70

What are the first and third quartiles? What is the 5 number summary?

Answer

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value *is* a whole number, we will find the mean of the 2nd and 3rd data values: $(60+62)/2 = 61$, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: $(67+69)/2 = 68$, so Q3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

What are the 5-number summaries for each of the previous 2 examples?

Answer

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

More about each set of women's heights is in the following videos.

Watch this video online: <https://youtu.be/00iQvPOOUu4>

Watch this video online: <https://youtu.be/x73G2Nep05g>

Returning to our quiz score data: in each case, the first quartile locator is $0.25(10) = 2.5$, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

A video walkthrough of this example is available below.

Watch this video online: <https://youtu.be/uifLbZKPUDU>

Try It Now

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

\$140 \$160 \$160 \$165 \$180 \$220 \$235 \$240 \$250 \$260 \$280 \$285
 \$285 \$285 \$290 \$300 \$300 \$305 \$310 \$310 \$315 \$315 \$320 \$320
 \$330 \$340 \$345 \$350 \$355 \$360 \$360 \$380 \$395 \$420 \$460 \$460

Example

Returning to the household income data from earlier in the section, create the five-number summary.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Answer

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was \$35 thousand. We can see in the table that the minimum income is \$15 thousand, and the maximum is \$50 thousand.

To find Q1, we calculate the locator: $L = 0.25(100) = 25$. This is a whole number, so Q1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of \$15, so

Values 1 to 6 are \$15 thousand

The next 8 data values are \$20, so

Values 7 to $(6+8)=14$ are \$20 thousand

The next 11 data values are \$25, so

Values 15 to $(14+11)=25$ are \$25 thousand

The next 17 data values are \$30, so

Values 26 to $(25+17)=42$ are \$30 thousand

The 25th data value is \$25 thousand, and the 26th data value is \$30 thousand, so Q1 will be the mean of these: $(25 + 30)/2 = \$27.5$ thousand.

To find Q3, we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are \$35, so

Values 43 to $(42+19)=61$ are \$35 thousand

The next 20 data values are \$40, so

Values 61 to $(61+20)=81$ are \$40 thousand

Both the 75th and 76th data values lie in this group, so Q3 will be \$40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

This example is demonstrated in this video.

Watch this video online: <https://youtu.be/ECOeeDrUxpo>

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between \$40 thousand and \$50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

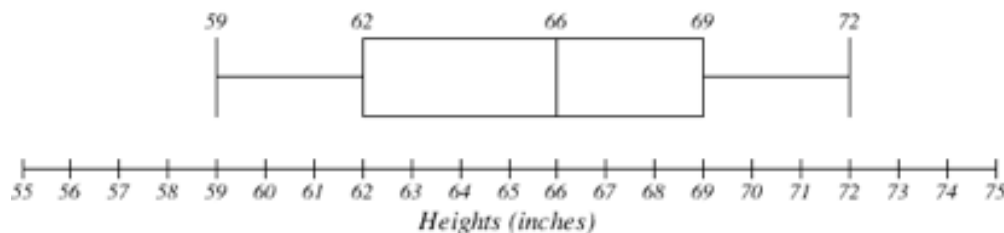
Box plot

A **box plot** is a graphical representation of a five-number summary.

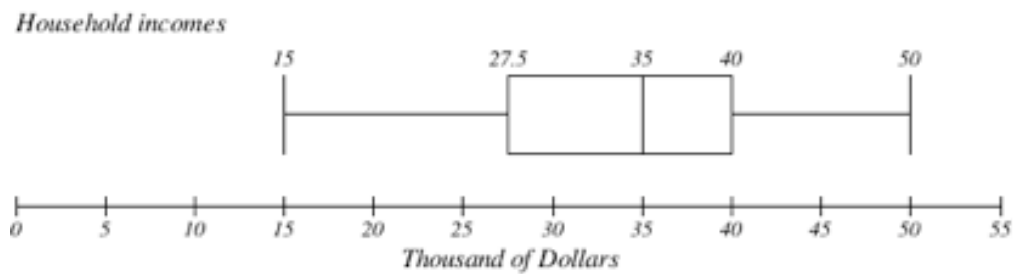
To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

Example

The box plot below is based on the 9 female height data with 5 number summary: 59, 62, 66, 69, 72.



The box plot below is based on the household income data with 5 number summary: 15, 27.5, 35, 40, 50



Box plot creation is described further here.

Watch this video online: <https://youtu.be/s4SPGFIMBMU>

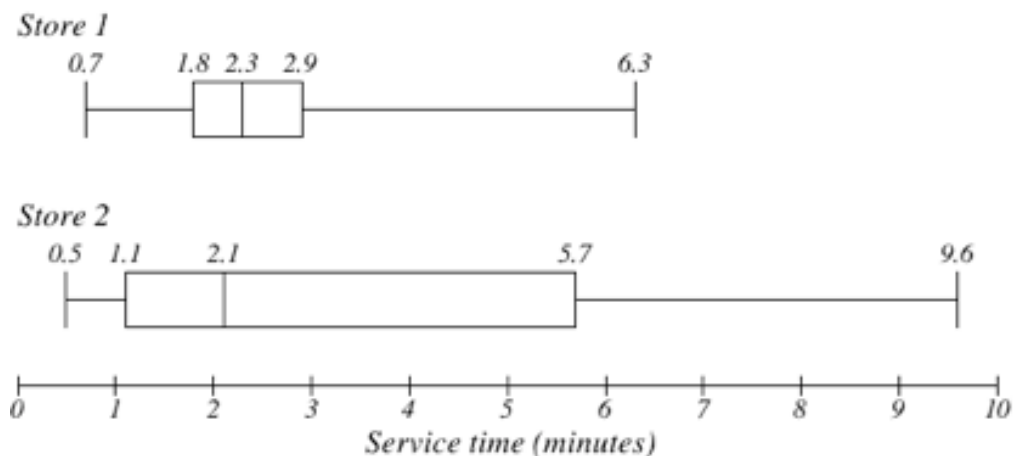
Try It Now

Create a box plot based on the textbook price data from the last Try It Now.

Box plots are particularly useful for comparing data from two populations.

Example

The box plot of service times for two fast-food restaurants is shown below.



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

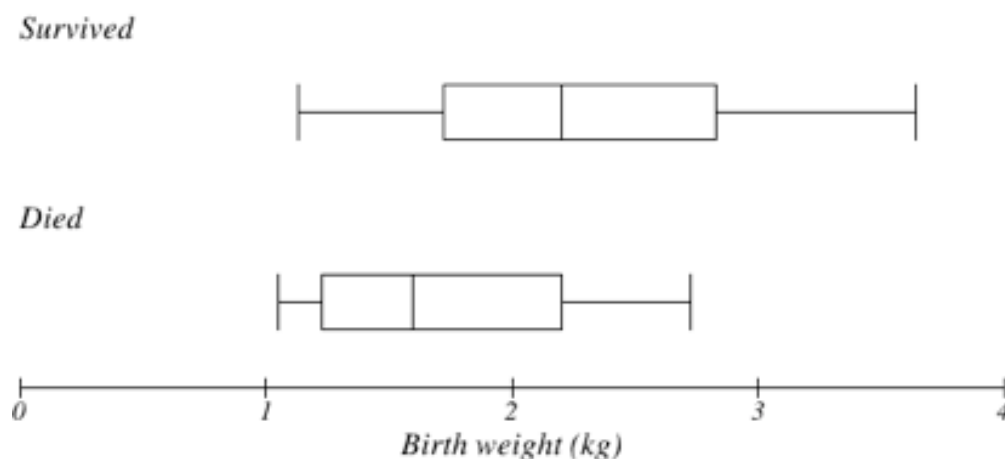
Which store should you go to in a hurry?

Answer

That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

The box plot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS) (Note: van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic

respiratory distress syndrome. Arch. Disease in Childhood, 48, 249–255. As quoted on <http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296§ion=1.1.3>). The box plot is separated to show the birth weights of infants who survived and those that did not.



Comparing the two groups, the box plot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

The following video analyzes the examples above.

Watch this video online: <https://youtu.be/eUkgf-2NVO8>

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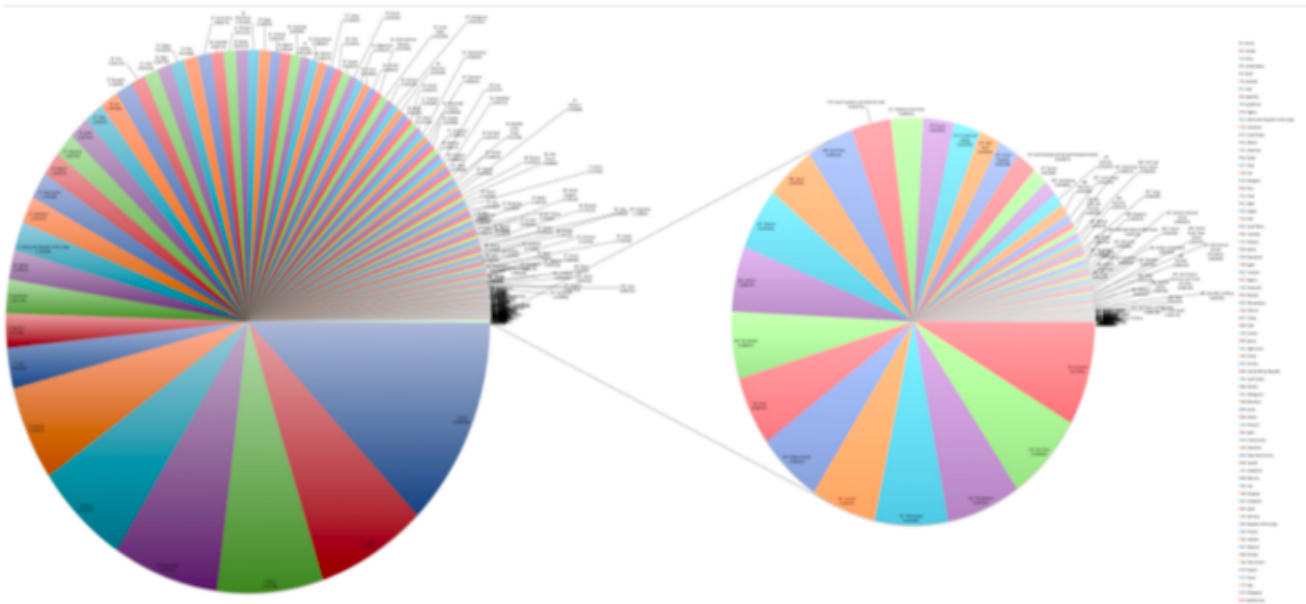
PUTTING IT TOGETHER: DESCRIBING DATA

The following is excerpted from “The Trials and Tribulations of Data Visualization for Good” by Jake Porway.

The trials and tribulations of data visualization for good

"I love big data. It's got such potential for storytelling." At [DataKind](#), we hear some version of this narrative every week. As more and more social organizations dip their toes into using data, invariably the conversation about data visualization comes up. There is a growing feeling that data visualization, with its combination of "engaging visuals" and "data-driven interactivity", may be the magic bullet that turn opaque spreadsheets and dry statistics into funding, proof, and global action.

However, after four years of applying data-driven techniques to social challenges at DataKind, we feel that data visualization, while it does have an important place in our work, is a mere sliver of what it takes to work with data. Worse, the ubiquity of data visualization tools has lead to a wasteland of confusing, ugly, and sometimes unhelpful pie charts, word clouds, and worse.



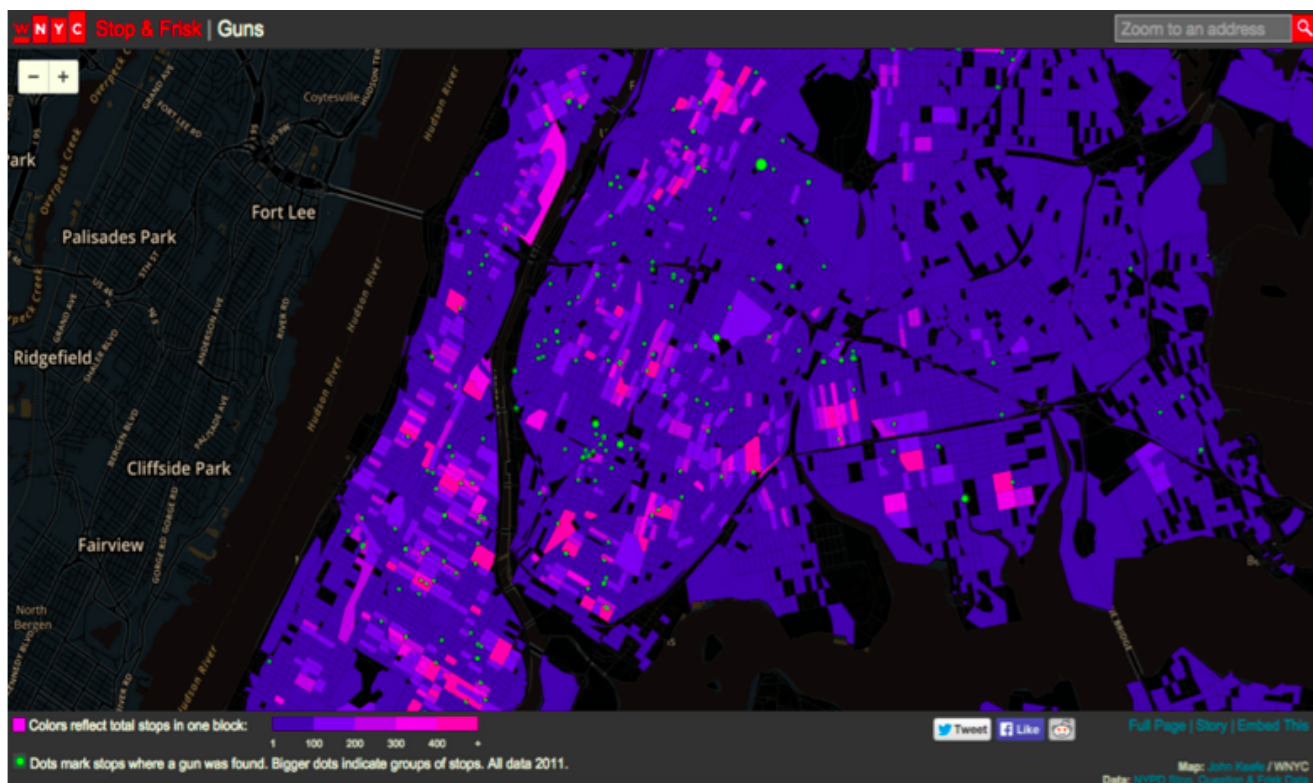
Ugh.

The challenge is that data visualization is not an end-goal, it is a process. It is often the final step in a long manufacturing chain along which data is poked, prodded, and molded to get to that pretty graph. Ignoring that process is at best misinformed, and at worst destructive.

Let me show you an example: In New York City, we had a very controversial program called [Stop and Frisk](#) that allowed police officers to stop people on the street they felt were a potential threat in an attempt to find and reclaim illegal weapons.

After a [Freedom of Information Act \(FOIA\)](#) request by the [New York Civil Liberties Union \(NYCLU\)](#) resulted in the [New York Police Department \(NYPD\)](#) releasing all of their Stop and Frisk data publicly, people flocked to the data to independently pick apart how effective the program was.

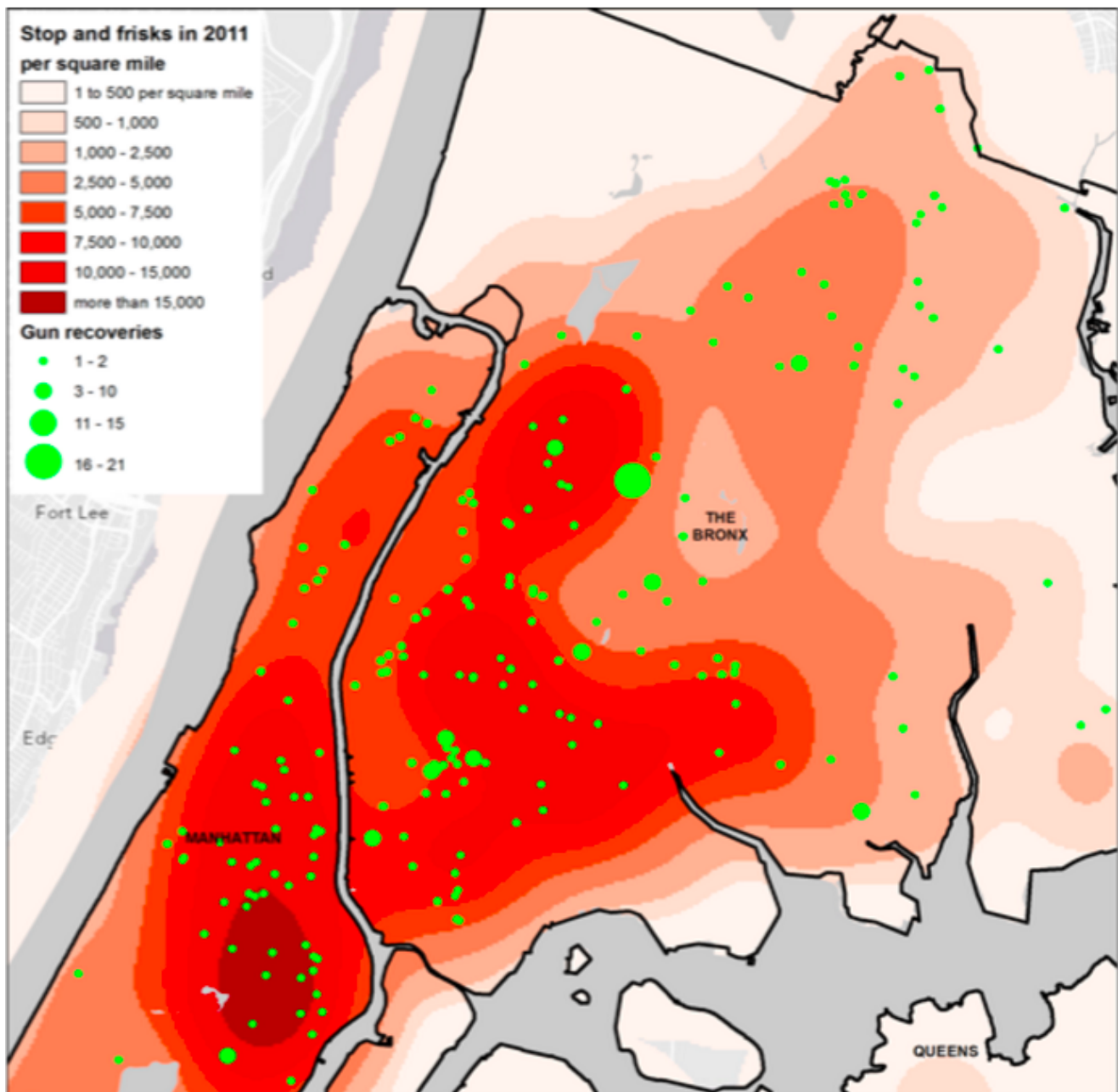
The figure below comes from WNYC, a public radio station located in New York City. Here they've shaded each city block brighter pink the more stops and frisks occurred there. The green dots on the map indicate where guns were found. What the figure shows is that the green dots do not appear as close to the hot pink squares as one would believe they should. The implication, then, is that Stop and Frisk may not actually be all that effective in getting guns off the street.



But then a citizen journalist created *this* map of the same data.



By simply changing the shading scheme slightly he notes that this map makes the green dots look much closer to the hot pink squares. In fact, he goes further to remove the artificial constraints of the block-by-block analysis and smooths over the whole area in New York, resulting in a map where those green dots stare unblinkingly on top of the hot-red stop and frisk regions.



The argument this author makes visually is that Stop and Frisk *does* in fact work.

So who's right here? Well both of them. And neither of them. These pictures are just that – pictures. Though they “use” data, they are not science. They are not analyses. They are mere visuals.

When data visualization is used simply to show alluring infographics about whether people like Coke or Pepsi better, the stakes of persuasion like this are low. But when they are used as arguments for or against public policy, the misuse of data visualization to persuade can have drastic consequences. **Data visualization without rigorous analysis is at best just rhetoric and, at worse, incredibly harmful.**

“Data for Humans vs. Data for Machines”

The fundamental challenge underlying this inadvertently malicious use of data comes, I believe, from a vagueness in terminology. When people crow about “the promise of data”, they are often describing two totally different activities under the same umbrella. I’ve dubbed these two schools of thought “data for humans” vs. “data for machines”.

Data for Humans: The most popular use of data, especially in the social sector, places all of the emphasis on the data itself as the savior. The idea is that, if we could just show people more data, we could prove our impact, encourage funding, and change behavior. Your bar charts, maps, and graphs pointing-up-and-to-the-right all fall squarely into this category. In fact almost all data visualization falls here, relying on the premise that showing a decisionmaker some data about the past will be all it takes to drive future change.

Unfortunately, while I believe data is a necessary part of this advocacy work, it is never sufficient by itself. The challenge with using “data for humans” is threefold:

1. **Humans don’t make decisions based on data**, at least not alone. Plato once said “Human behavior flows from three main sources: desire, emotion, and knowledge.” I want to believe he listed those aspects in that order intentionally. [Study](#) after [study](#) has shown that humans rationalize beliefs with data, not vice versa. If behavior change were driven by data and graphs alone, we would be 50 years into a united battle against climate change. Conversely, we will leap to conclusions from data visualizations that “feel” right, but are not rigorously tested, like the conclusions from the Stop and Frisk images above.
2. **The public still treats data and data visualization as “fact” and “science”**. I believe the public has gained enough visual literacy to question photojournalists or documentary filmmakers’ motives, aware that theirs is an auteur behind the final piece that intends for us to walk away with their chosen understanding. We have yet to bring that same skepticism to data visualization, though we need to. The result of this illiteracy is that we are less critical of graphs and charts than written arguments because the use of data gives the sense that “fact” or “science” is at work, even if what we’re doing is little more than visually bloviating.
3. **The data or visualization you see at the end of the road is opaque to interrogation**. It is difficult, if not impossible to know where that “58%” statistic or that flashy bar graph came from, grinning up at you from the page. Because we don’t have ways to know how the data was collected, manipulated, and designed, we can’t answer any of the questions we might want to raise above. If point 2 means we need to treat data visualization as photojournalism, then this point implores us to go further to requiring forensic photographers in this work.

Data for Machines: For these reasons, DataKind specializes in projects focusing on what we refer to as “Data for Machines”. The promise of abundant data is not that we can show people more data, but that we can take advantage of computers, algorithms, and rigorous statistical methodologies to learn from these new datasets. The data is not the end goal, it is the raw resource we use to fuel computer systems that can learn from this information and, in many cases, even predict what is likely to happen in the future.

For example, instead of engaging in the Stop and Frisk gallery debate above, DataKind volunteers loaded the NYPD data into computers and created statistical models to rigorously test whether or not racial discrimination was occurring disproportionately in different parts of the city. While the models needed further evaluation, this analysis shows how data should be used. People shouldn’t try to draw conclusions from pictures of data – we’re notoriously bad at that as humans – we should be building models and using scientific methods to learn from data.

Celebrating Visualization

No surprise, creating data visualization well simply entails designing in a way that leads people to make scientific conclusions themselves.

There are many examples of [insightful](#), [persuasive](#), and [downright clever](#) data visualizations, but perhaps one of the best visualization practices I know of is to turn the idea of visualization on its head. Data visualization is incredibly good for allowing one to ask questions, not answer them. The huge amount of data that we have available to us now means that we need visual techniques just to help us make sense of what we need to try to make sense of.

So where do we go from here?

First off, you can boycott the tyranny of pie charts and word clouds, rail against those three pitfalls, and share these last two examples far and wide. But I think we can also all go out and start thinking about how data can truly be used to its fullest advantage. Aside from just using “data for machines,” the best data visualization should raise questions and inspire exploration, not just sum up information or try to tell us the answer. Today we have more information than ever before and we have a new opportunity to use it to mobilize others,

provided we do so with sensitivity. Now, more than ever, we need to all be out there on the front lines looking beyond data visualization as merely a way to satisfy our funders' requirements and instead looking at data as a way to ask deep questions of our world and our future.

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PROBABILITY

WHY IT MATTERS: PROBABILITY

According to the news, the lottery jackpot is climbing by the hour. Long lines of dreamers are forming wherever lottery tickets are sold. While you don't usually buy lottery tickets, it is getting tempting. Just imagine what you could do with \$100 million dollars. Perhaps you could retire early or never even go to work. Maybe buy a rare fancy car. All you need to do is pick the correct numbers, and the jackpot is all yours!



It sounds easy enough; just six simple numbers. But how likely are you to win? And could you increase the likelihood of winning by purchasing more lottery tickets?



To answer these questions, you need to know about permutations and combinations. So learn about them as you complete this module, and then we'll return to the lottery at the end. Then you'll be able to decide whether you want to stand in line to purchase a ticket.

Learning Objectives

Computing the Probability of an Event

- Describe a sample space and simple and compound events in it using standard notation
- Calculate the probability of an event using standard notation
- Calculate the probability of two independent events using standard notation
- Recognize when two events are mutually exclusive
- Calculate a conditional probability using standard notation

Applications With Probability

- Compute a conditional probability for an event
- Use Baye's theorem to compute a conditional probability
- Calculate the expected value of an event

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INTRODUCTION: COMPUTING THE PROBABILITY OF AN EVENT

Learning Objectives

The learning outcomes for this section include:

- Describe a sample space and simple and compound events in it using standard notation
- Calculate the probability of an event using standard notation
- Calculate the probability of two independent events using standard notation
- Recognize when two events are mutually exclusive
- Calculate a conditional probability using standard notation

Probability is the likelihood of a particular outcome or event happening. Statisticians and actuaries use probability to make predictions about events. An actuary that works for a car insurance company would, for example, be interested in how likely a 17 year old male would be to get in a car accident. They would use data from past events to make predictions about future events using the characteristics of probabilities, then use this information to calculate an insurance rate.

In this section, we will explore the definition of an event, and learn how to calculate the probability of it's occurrence. We will also practice using standard mathematical notation to calculate and describe different kinds of probabilities.



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BASIC CONCEPTS

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes.



We begin with some terminology.

Events and Outcomes

- The result of an experiment is called an **outcome**.
- An **event** is any particular outcome or group of outcomes.
- A **simple event** is an event that cannot be broken down further
- The **sample space** is the set of all possible simple events.

Example

If we roll a standard 6-sided die, describe the sample space and some simple events.

Answer

The sample space is the set of all possible simple events: $\{1,2,3,4,5,6\}$
Some examples of simple events:

- We roll a 1
- We roll a 5

Some compound events:

- We roll a number bigger than 4
- We roll an even number

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally-likely outcomes}}$$

Example

If we roll a 6-sided die, calculate

1. $P(\text{rolling a 1})$
2. $P(\text{rolling a number bigger than 4})$

Answer

Recall that the sample space is $\{1,2,3,4,5,6\}$

1. There is one outcome corresponding to “rolling a 1,” so the probability is $\frac{1}{6}$
2. There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

This video describes this example and the previous one in detail.

Watch this video online: <https://youtu.be/37P01dt0zsE>

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Answer

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try It Now

At some random moment, you look at your clock and note the minutes reading.

- a. What is probability the minutes reading is 15?

b. What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Answer

There are 52 cards in the deck and 4 Aces so $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event.

The largest possible probability is 1 – if all possible outcomes correspond with the event.

This video demonstrates both this example and the previous cherry example on the page.

Watch this video online: https://youtu.be/EBqj_R3dzd4

Certain and Impossible events

- An impossible event has a probability of 0.
- A certain event has a probability of 1.
- The probability of any event must be $0 \leq P(E) \leq 1$

Try It Now

Visit this page in your course online to practice before taking the quiz.

In the course of this section, if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.

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TYPES OF EVENTS

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = 1/6$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$



Complement of an Event

The **complement** of an event is the event “ E doesn’t happen”

- The notation \bar{E} is used for the complement of event E .
- We can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$
- Notice also that $P(E) = 1 - P(\bar{E})$

Example

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Answer

There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.

The probability of *not* drawing a heart is the complement: $P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$

This situation is explained in the following video.

Watch this video online: <https://youtu.be/RnljiW6ZM6A>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Probability of two independent events

Example

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Answer

We could list all possible outcomes: {H1,H2,h2,H4,H5,H6,T1,T2,T3,T4,T5,T6}.

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example contained two *independent* events. Getting a certain outcome from rolling a die had no influence on the outcome from flipping the coin.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example

Are these events independent?

1. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
2. The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

3. You draw a card from a deck, then draw a second card without replacing the first.

Answer

1. The probability that a head comes up on the second toss is $1/2$ regardless of whether or not a head came up on the first toss, so these events are independent.
2. These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
3. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for independent events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Answer

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$.

Examples of joint probabilities are discussed in this video.

Watch this video online: <https://youtu.be/6F17WLp-EL8>

Try It Now

Visit this page in your course online to practice before taking the quiz.

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Answer

Here, there are still 12 possible outcomes: {H1, H2, h2, H4, H5, H6, T1, T2, T3, T4, T5, T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, h2, H4, H5, H6, T6), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

$P(A \text{ or } B)$

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Answer

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$.

Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

See more about this example and the previous one in the following video.

Watch this video online: <https://youtu.be/klbPZeH1np4>

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Answer

Half the cards are red, so $P(\text{red}) = \frac{26}{52}$

There are four kings, so $P(\text{King}) = \frac{4}{52}$

There are two red kings, so $P(\text{Red and King}) = \frac{2}{52}$

We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Try It Now

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Example

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

1. Has a red car *and* got a speeding ticket
2. Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Answer

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

$$P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket}) \\ = \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}.$$

This table example is detailed in the following explanatory video.
Watch this video online: <https://youtu.be/HWrGoM1yRaU>

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CONDITIONAL PROBABILITY

In the previous section we computed the probabilities of events that were independent of each other. We saw that getting a certain outcome from rolling a die had no influence on the outcome from flipping a coin, even though we were computing a probability based on doing them at the same time.

In this section, we will consider events that *are* dependent on each other, called **conditional probabilities**.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$$P(B | A)$$

This is read as “the probability of B given A ”

For example, if you draw a card from a deck, then the sample space for the next card drawn has changed, because you are now working with a deck of 51 cards. In the following example we will show you how the computations for events like this are different from the computations we did in the last section.

Example

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

Answer

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the “condition” is that the first card is an ace. Symbolically, we write this as:

$P(\text{ace on second draw} \mid \text{an ace on the first draw})$.

The vertical bar “|” is read as “given,” so the above expression is short for “The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw.” What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Conditional Probability Formula

If Events A and B are not independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

Example

If you pull 2 cards out of a deck, what is the probability that both are spades?

Answer

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

1. has a speeding ticket *given* they have a red car
2. has a red car *given* they have a speeding ticket

Speeding ticket	No speeding ticket	Total
-----------------	--------------------	-------

Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Answer

1. Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so $P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$
2. Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so $P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25$.

Notice from the last example that $P(B \mid A)$ is **not** equal to $P(A \mid B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

View more about conditional probability in the following video.

Watch this video online: <https://youtu.be/b6tstekMIb8>

Example

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Answer

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$.

The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is $P(A) + P(B) - P(A \text{ and } B)$. In this problem, $P(A \text{ and } B) = 0$ since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{102} + \frac{1}{102} = \frac{2}{102}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{2}{102}$.

These two playing card scenarios are discussed further in the following video.

Watch this video online: https://youtu.be/ngyGsgV4_0U

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results.

Find

1. $P(\text{not pregnant} \mid \text{positive test result})$
2. $P(\text{positive test result} \mid \text{not pregnant})$

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

Answer

1. Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant. $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067$.
2. Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

See more about this example here.

Watch this video online: <https://youtu.be/LH0cuHS9Ez0>

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INTRODUCTION: APPLICATIONS WITH PROBABILITY

Learning Objectives

Applications With Probability

- Compute a conditional probability for an event
- Use Baye's theorem to compute a conditional probability
- Calculate the expected value of an event

In the next section, we will explore more complex conditional probabilities and ways to compute them. Conditional probabilities can give us information such as the likelihood of getting a positive test result for a disease without actually having the disease. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. If you were to get a positive test result, knowing the likelihood of getting a false positive can guide you to get a second opinion.

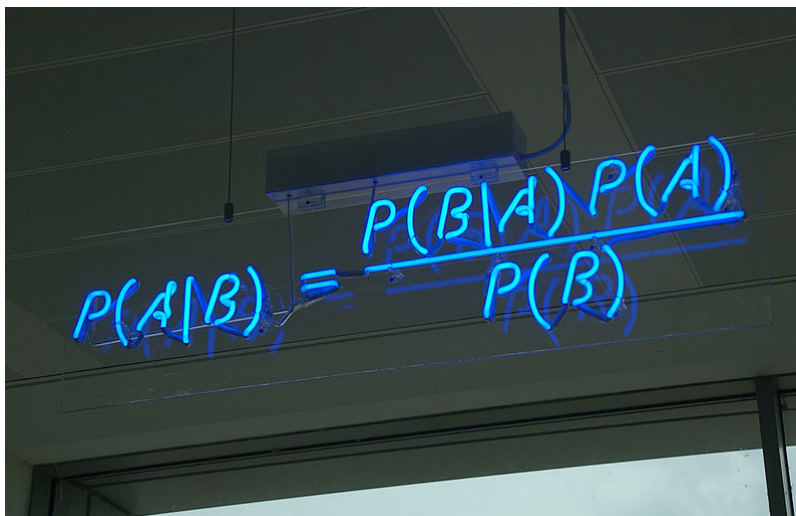
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BAYES' THEOREM

In this section we concentrate on the more complex conditional probability problems we began looking at in the last section.


$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$

For example, suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece as an example.

Example

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: $1/1000$. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write $P(\text{disease})=0.001$.)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say $P(\text{positive} | \text{disease})=1$.)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that $P(\text{positive} | \text{no disease})=0.05$.)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute $P(\text{disease} | \text{positive})$. We already know that $P(\text{positive} | \text{disease})=1$, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about $1/1000$ of all people are afflicted with the disease, $1/1000$ of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect $(0.05)(999)=49.95$ (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing $P(\text{disease} | \text{positive})$. There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$P(\text{disease} | \text{positive}) \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and $(0.05)(99,900)=4995$ test positive but do not have the disease, so the exact probability of having the disease if you test positive is

$$P(\text{disease} | \text{positive}) \approx \frac{100}{5095} \approx 0.0196$$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide. This example is worked through in detail in the video here. Watch this video online: <https://youtu.be/hXevfqsBino>

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\bar{A})P(B|\bar{A})}$$

In our earlier example, this translates to

$$P(\text{disease}|\text{positive}) = \frac{P(\text{disease})P(\text{positive}|\text{disease})}{P(\text{disease})P(\text{positive}|\text{disease}) + P(\text{nodisease})P(\text{positive}|\text{nodisease})}$$

Plugging in the numbers gives

$$P(\text{disease}|\text{positive}) = \frac{(0.001)(1)}{(0.001)(1) + (0.999)(0.05)} \approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

Example

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Answer

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test

positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{disease}|\text{positive}) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{disease}|\text{positive}) = \frac{(0.02)(0.90)}{(0.02)(0.90) + (0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

View the following for more about this example.

Watch this video online: https://youtu.be/_c3xZvHto3k

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COUNTING

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich

breadsticks + quiche

breadsticks + fajita

breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

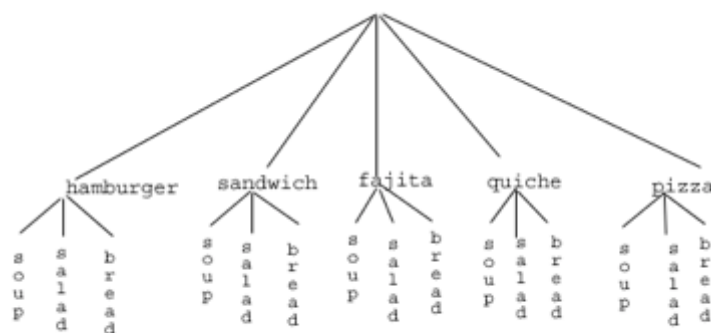
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	sandwich	quiche	fajita	pizza
soup	soup+burger				
salad	salad+burger				
bread	etc				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a “tree” diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5)

to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$. This is sometimes called the multiplication rule for probabilities.

Example

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Answer

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Answer

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possibilities.

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Answer

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3 , which is much more compact.

Basic counting examples from this section are described in the following video.

Watch this video online: <https://youtu.be/fROqcu-ekkw>

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Answer

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code word with the letters MATH.

In this example, we needed to calculate $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

Factorial

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$$

Try It Now

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Example

How many ways can five different door prizes be distributed among five people?

Answer

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

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Now we will consider some slightly different examples.

Example

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Answer

Using the Basic Counting Rule, there are 25 choices for the person who receives the \$100 certificate, 24 remaining choices for the \$25 certificate and 23 choices for the \$10 certificate, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways in which the prizes can be awarded.

Example

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Answer

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the \$100 prize, or a different goldmedalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a \$10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

Factorial examples are worked in this video.

Watch this video online: https://youtu.be/9maoGi5fd_M

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often; we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$${}_nP_r = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$${}_nP_r = \frac{n!}{(n-r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Answer

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

Example

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Answer

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$.
View this video to see more about the permutations examples.
Watch this video online: <https://youtu.be/xlyX2UJMJQI>

Try It Now

How many 5 character passwords can be made using the letters A through Z

- if repeats are allowed
- if no repeats are allowed

Visit this page in your course online to practice before taking the quiz.

Combinations

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Answer

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:
ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all. We can just use permutations! So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $13,800/6 = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is not important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by $\frac{{}_nP_r}{{}_rP_r}$, and we could use this formula for computation. However this situation arises so frequently that we attach a special

notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* where the *order of selection* is **not** important.

Combinations

$${}_nC_r = \frac{{}_nP_r}{r!}$$

We say that there are ${}_nC_r$ **combinations** of size r that may be selected from among n choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Answer

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection* is **not** important there are ${}_{35}C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$ combinations.

View the following for more explanation of the combinations examples.

Watch this video online: <https://youtu.be/W8kd4YosbzE>

Try It Now

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try It Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Answer

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are $15C10 = 3003$ ways to choose the 10 Republicans and $14C9 = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$ possible ways of selecting the members of the Defense Subcommittee.

This example is worked through below.

Watch this video online: <https://youtu.be/Xqc2sdYN7xo>

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PROBABILITY USING PERMUTATIONS AND COMBINATIONS

We can use permutations and combinations to help us answer more complex probability questions.

Example

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Answer

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation $10P4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{10P4}{10^4} = \frac{5040}{10000} = 0.504$

Try It Now

Visit this page in your course online to practice before taking the quiz.

Example

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Answer

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of \$1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Answer

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$. So the probability of winning the second prize is.

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

The previous examples are worked in the following video.

Watch this video online: https://youtu.be/b9LFbB_aNAo

Example

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Answer

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, $52C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be $4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be $48C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be $4C_1 \cdot 48C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Answer

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

View the following for further demonstration of these examples.

Watch this video online: <https://youtu.be/RU3e3KTkjoA>

Try It Now

Visit this page in your course online to practice before taking the quiz.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Answer

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves “What is the alternative to having at least one shared birthday?” In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to “at least one” is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let’s imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let’s ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person’s, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{nosharedbirthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let’s make our group a bit bigger.

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Answer

Continuing the pattern of the previous example, the answer should be

$$P(\text{sharedbirthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{sharedbirthday}) = 1 - \frac{365 P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Answer

Here we can calculate

$$P(\text{sharedbirthday}) = 1 - \frac{365 P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

The birthday problem is examined in detail in the following.

Watch this video online: https://youtu.be/UUmTfiJ_0k4

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts.

Try It Now

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

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EXPECTED VALUE

Repeating Procedures Over Time

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.



Example

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Answer

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $-\$2/\$38 \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$.

The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
-\$1	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

Expected Value

- **Expected Value** is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try It Now

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Example

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Answer

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$
\$999	$\frac{252}{12271512}$
-\$1	$1 - \frac{253}{12271512} = \frac{12271259}{12271512}$

The expected value, then is:

$$(\$999,999) \cdot \frac{1}{12271512} + (\$999) \cdot \frac{252}{12271512} + (-\$1) \cdot \frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose \$1. View more about the expected value examples in the following video.

Watch this video online: <https://youtu.be/pFzgxGVltS8>

Try It Now

Visit this page in your course online to practice before taking the quiz.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Try It Now

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. What is the expected value of this game? Would you play?

Visit this page in your course online to practice before taking the quiz.

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year. (Note: According to the estimator at <http://www.numericalexample.com/index.php?view=article&id=91>) An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

Answer

The probabilities and outcomes are

Outcome	Probability of outcome
\$100,000 – \$275 = \$99,725	0.00242
-\$275	$1 - 0.00242 = 0.99758$

The expected value is $(\$99,725)(0.00242) + (-\$275)(0.99758) = -\$33$.

The insurance applications of expected value are detailed in the following video.

Watch this video online: <https://youtu.be/Bnai8apt8vw>

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

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PUTTING IT TOGETHER: PROBABILITY

The lottery jackpot has continued to climb as you completed this module. Now it is time to determine how likely you are to win.

Let's first assume that you not only need to pick six specific numbers from 1 – 49, but you need to pick them in the correct order. If this is the case, you know you need to use a permutation to figure out the size of the sample space.

$$P(n, r) = \frac{n!}{(n-r)!}$$

In this case, n is the possible numbers, which is 49, and r is the number of choices you make, which is 6.

$$P(49, 6) = \frac{49!}{(49-6)!}$$

$$P(49, 6) = \frac{49!}{43!} = 10,068,347,520$$

This tells you that there is one way out of about 10 billion to win; your chances are not good at all.



Fortunately, most lottery winnings do not depend on order so you can use a combination instead.

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

$$C(49, 6) = \frac{49!}{6!(49-6)!}$$

$$C(49, 6) = \frac{49!}{6!(43)!}$$

$$C(49, 6) = \frac{49!}{6!(43)!} = 13,983,816$$

Notice that the sample space has been greatly reduced from about 10 billion to about 14 million. So the likelihood of you winning is much greater than before, but still very slim.

What would happen to your chances of winning if you bought more than one ticket? Suppose you bought 100 tickets and chose a different combination of six numbers on each ticket. You could compare the number of tickets to sample space to determine your probability.

$$\frac{100}{14 \text{ million}} = 0.0000071 = 0.00071\%$$

That's much less than a 1% chance of winning. Still not very good. So suppose you gather up some cash and buy 1,000 tickets.

$$\frac{1,000}{14 \text{ million}} = 0.000071 = 0.0071\%$$

Now you are out \$1000, assuming each ticket costs \$1, and your chances are still less than a 1% chance.

Okay, maybe you are ready to go for broke. You and a group of friends gather your funds to purchase 1 million tickets.

$$\frac{1 \text{ million}}{14 \text{ million}} = 0.071 = 7.1\%$$

So even after purchasing 1 million tickets, which might cost \$1 million, your probability of winning the big jackpot is only about 7%. To raise your probability to just 50%, you would have to purchase 7 million tickets. It's up to you do decide how lucky you feel. Maybe just buy one ticket and see what happens. Good luck!

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